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ABSTRACT

This collection of nine papers, prepared for a conference held at Northwestern University in 1978, presents varied perspectives on applied problem solving. Assessing applied problem solving, planning for interest and motivation, developing a theory, reviewing research findings, considering learning disabilities, analyzing through information processing, designing instruction, trends in research, and models for applied problem solving are presented. (Author/MK)

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# APPLIED MATHEMATICAL PROBLEM SOLVING

Edited by

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# FOREWORD

This collection of papers, prepared for a conference held at Northwestern University in 1978, presents varied perspectives on applied problem solving. Assessing applied problem solving, planning for interest and motivation, developing a theory, reviewing research findings, considering learning disabilities, analyzing through information processing, and designing instruction are all considered, as well as trends in research and models for applied problem solving.

A wealth of information is thus provided: we hope this publication will serve to extend current thinking about what should be taught about applied problem solving -- and how it should be taught.

Marilyn N. Suydam
ERIC/SMEAC

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# Introduction: A Focus on Applied Mathematical Problem Solving

Richard Lesh
Diane Mierkiewicz Editors
Mary Kantowski

The papers in this monograph developed out of a conference on applied mathematical problem solving held at Northwestern University in January 1978. The purpose of the Northwestern meeting, and the purpose of the papers in this monograph, is to review a variety of perspectives concerning the general question, "What is it, beyond having a concept, that enables an average ability student to use the idea in real situations?"

In the United States, results of recent National Assessments of Educational Progress in mathematics suggest that "Johnny can add; computation with whole numbers is far from a lost art" (Carpenter et al., 1975, p. 457). However, knowing how to compute does not ensure that a person will know when to compute, which operation to use in a particular situation, or how to use an answer once it is obtained. Performance on NAEP items seemed most discouraging on items where youngsters needed to use number ideas to answer questions involving measurement or consumer oriented situations.

> ...The implications for needed improvement in mathematics programs are abundant in these data. As a whole, these age groups need to develop more problem solving skills. Even such fundamental habits as checking the correctness or reasonableness of a result, or making an estimate, seem to be lacking. (p. 470)

In general, "being able to use a concept" involves something more than simply "having the concept." Getting an idea into a youngster's head does not guarantee that the new idea will be integrated with other ideas that are already understood, that situations will be recognized in which the idea is relevant, nor that the library-type "look up" skills will be available to retrieve related ideas when they are needed. "Being able to use an idea" may also involve certain problem solving processes in addition to those needed to demonstrate the simple attainment of the concept. However, the processes that are needed to use a mathematical idea in real situations are not necessarily the type that have been discussed by Polya (1957), Landa (1974, 1976), Wickelgren (1974), Davis (1973), or other problem solving theorists who are popular among mathematics educators.

Research and instruction on problem solving have made little progress toward dealing with the issue of what it is, beyond having an idea, that allows a normally intelligent person to use the idea to deal with the math related problems in everyday situations. Applied problem solving processes constitute an important part of the basic skills that are needed for mathematical literacy among average citizens; yet, they have not been emphasized

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by spokespersons for either "basic skills" or "problem solving."

What is appled mathematical problem solving? Applied problem solving occurs when ordinary people attempt to solve real problems in real (or at least realistic) situations. Unfortunately, however, most information about problem solving processes has come from situations involving older students, exceptionally bright students, individual students working in isolation (often in artificial laboratory situations), or situations involving highly contrived work problems, proofs, or mathematical puzzles which involve underlying ideas that are of questionable mathematical worth. Elementary or middle school children, average (or below average) ability students, and applied problem solving processes have been neglected. For this reason, the "problem solving" processes educators discuss often seem inaccessible to younger children or less gifted students, and applied problem solving processes, like modeling, have been ignored.

The papers in this monograph will describe some promising directions for future problem solving research and instruction, and they will focus on ideas and processes that are accessible to average ability middle school or elementary school children when they try to solve real problems involving substantive mathematical ideas in realistic situations. The articles included here should be quite useful to anyone wishing to understand issues related to R & D efforts on the forefront of this field.

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# Applied Problem Solving as a School Emphasis: An Assessment and Some Recommendations

Max S. Bell
University of Chicago

Given the obvious gulf between what we, as mathematics educators, persistently say we want to do about "problem solving" in school mathematics and our actual results, I began some months ago to examine reports of research on problem solving for clues about how to narrow that gulf. But, in that literature, I found very little that is applicable to improving instruction even for conventional "word problems" and practically nothing related to "applied problem solving." Hence, instead of the review of research results that I had intended for this presentation, I will undertake an examination of why it is that despite such earnest intentions and so much research, we are still at square one in our knowledge of how to help children become better solvers of applied problems.

I may be too gloomy in concluding that years of work and vast amounts of earnest effort have left us nearly bereft of practical instructional guidance, but here in brief outline is the argument I will make for that conclusion:

1. "Problem solving" means so many things to so many people that some more precise formulation of the phrase is essential in order to discuss it at all. But there is not that ambiguity in many of the prescriptions for "reform" of school mathematics instruction. In them, "problem solving" consistently means something like applied problem solving as addressed by this conference; that is, the solving of "real problems with real data."

2. If applied problem solving is our concern, such evidence as we have from assessments of the net effects of school instruction indicates that we are anything but effective.

3. That isn't so very surprising since real problems with real data essentially do not exist in the textbooks that dominate school mathematics instruction. Hence, most children simply do not encounter applied problem solving in their school work, and specific instruction in problem solving of any sort is probably rare.

4. We cannot count on enlightenment about applied problem solving from those (mainly psychologists) who do research on human problem solving. Such research is voluminous but it is so chaotic and its concerns are generally so remote from applied problem solving that it seems to me to be nearly useless as a guide to school instruction.

5. Hence, instruction in applied problem solving is very likely a

problem we mathematics educators must sort out for ourselves. Unfortunately, mathematics education research in problem solving seldom deals with real problems with real data, though there is quite a lot of research focused on conventional problem solving and geometric proof.

6. Over the past few years very promising curriculum resources have been produced that would, properly used, support an emphasis on applied problem solving at all school levels. Many of these resources, especially for elementary school, come from science educators. But we have failed to take even the simplest and most obvious first step toward these materials being absorbed into day-to-day school work. It is not difficult to list some obvious things we have failed to do, and I will start such a list.

7. It is surely not too soon to undertake new research and development initiatives to end the neglect of applied problem solving in school work. I will attempt a preliminary list of research and development initiatives to that end.

In what follows, I will take up each of these points in order, not by way of tidy and complete scholarly arguments but in the spirit of brief outline and deliberate provocation. I believe that each such point can be supported at length, but there is not time today for such detail.

As I go through this survey of our disastrous neglect of something quite important, I want several things to be clearly understood. First, when I speak of some obvious shortcomings, I do not mean that overcoming them will necessarily be easy. Second, when I advocate something, even very strongly, I do not necessarily mean by that to denigrate some related enterprise; that is, relatively less important is not for me the same as unimportant. Third, when I say that something is presently irrelevant as a guide to better applied problem solving instruction, I do not mean to suggest that it is irrelevant in general. Please keep that in mind, especially as I discuss the research to date on human problem solving. Fourth, while I hope to make it very clear that we have hardly begun to make applied problem solving a viable school emphasis, it should be equally clear that it would be silly and fruitless to try to assign "blame" for that.

## Problem Solving, Applied Problem Solving, Real Problems with Real Data

The first obvious point to make is that "problem solving" means many things to many people. From this flows much of our confusion in school instruction. Yet what is meant by problem solving when that phrase is used in stating objectives for school mathematics instruction seems quite consistent within itself, as indicated by these statements spanning half a century.*

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* The brief search that yielded these and similar statements illustrated again the discouraging fact that virtually isomorphic statements of our

> Problem solving in school is for the sake of solving problems in life. Other things being equal, problems where the situation is real are better than problems where it is described in words. Other things being equal, problems which might really occur in a sane and reasonable life are better than bogus problems and mere puzzles...A better selection of problems will probably be secured if instead of searching for problems that conveniently apply specific topics we search for problems that are intrinsically worth learning to solve.... (Thorndike et al., 1923, p. 154)

> Since mathematics has proved indispensable for the understanding and the technological control not only of the physical world but also of the social structure, we can no longer keep silent about teaching mathematics so as to be useful. In educational philosophies of the past, mathematics often figures as the paragon of a disinterested science. No doubt it still is, but we can no longer afford to stress this point if this keeps our attention off the widespread use of mathematics and the fact that mathematics is needed not by a few people but virtually by everybody. (Freudenthal, 1968, p. 4)

> Mathematics is a lot of fun for a small number of individuals. For even a smaller number mathematics provides a profound aesthetic experience. If that were the whole story it would not be possible to justify the emphasis given to mathematics in our school programs. The real justification for teaching mathematics in our schools is that it is a useful subject and, in particular, that it helps in solving many kinds of problems. (Begle, 1979)

Many others over the years and virtually every group suggesting "reform" of school mathematics have been explicit in identifying what this conference calls applied problem solving as the ultimate (but not the only) justification for school mathematics instruction. I have already said I feel that means attention to real problems with real data, and I believe that is possible with youngsters of every school age. Yet the net burden of my remarks today will be that we appear to be confused and ineffectual with respect to applied problem solving in every phase of the curriculum and instructional process: what to do, how to do it, assessment of what we have done, and in research and development aimed at improvement of instruction.

## The Net Effects of School Mathematics Instruction

Our measures of what we accomplish in school mathematics are far from adequate but the accumulation of such evidence as we have indicates that

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* (continued from previous page) principal problems in mathematics education tend to appear again and again over the years, thus implicitly documenting our persistent failures in solving instructional problems.

most people eventually learn to do arithmetic but perhaps only half are able to use arithmetic, even for solving simple and relatively artificial word problems. One strong indication of that comes from the National Assessment of Educational Progress (NAEP, 1975a, 1975b). The NAEP findings are based on word problems that at best only dimly reflect what we might hope for in the way of applied problem solving and they generally ask only for uses of simple arithmetic. But since in the NAEP mathematics assessment even these rudimentary problems typically had 50% or fewer correct responses, it is not likely that we could expect better results with more genuine applied problem solving exercises. Hence, let us take it as given that there are widespread incapacities to deal with real problems with real data and search for some of the explanations of that.

## Applied Problem Solving in School Mathematics Textbooks

There is little doubt that except possibly for teachers themselves, the greatest single influence on school mathematics instruction is school mathematics textbooks. But such books do little with problem solving of any sort and even that which is done has nothing much to do with real problems with real data. I don't believe many people will dispute that and I assure you that I have checked it out beyond mere impression, at least with respect to typical first-year algebra textbooks and typical textbooks for grades one through five.*

That is not to say that there are not word problems with apparent real world context in those books, for as it happens, there are several hundred such in a typical book. But virtually all of those hundreds of word problems fit the Polya characterization of "one-rule-under-your-nose" (Polya, 1966). That is, in such problems students are explicitly directed to a single prescribed method. Very few are "problems which might really occur in a sane and reasonable life." Virtually none deal with real, or even realistic, data. Since such books largely determine the curriculum in most classrooms, it is very likely that in most classrooms applied problem solving is not taught in any meaningful way. Exceptions no doubt exist and we should search them out and treasure them.

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* An elementary school teacher colleague (Jean F. Bell) and I recently analyzed word problems in a widely used K-6 textbook series with respect to context, arithmetic operation or other solution method, size and kind of numbers used, realism, and the extent to which children would need to seek data, make decisions, or in other ways exercise problem solving skills beyond mere computing in a stereotyped way. Our results indicate that there is very little applied problem solving called for by such sets of problems. An earlier and similar study by myself for beginning algebra books was, if anything, even more discouraging (Bell, 1969).

Bell

## Applied Problem Solving and Psychological Research in Problem Solving

It is plausible to suppose that one interested in applied problem solving should begin by becoming familiar with the research on problem solving more generally. But in my own effort to do that, I found the sheer amount of literature on that subject to be nearly overwhelming, and the yield for one interested in school instruction disappointingly small. But I can at least offer a few suggestions with respect to starting points for one who might attempt a similar assessment of the psychological literature on human problem solving. For a simple and "popular" general overview of work on problem solving, Davis (1973) will serve. For a frequently cited treatment of the information processing way of approaching problem solving, I suggest Newell and Simon (1972), and for article length indications of that information processing point of view, Simon (1975a, 1975b), the latter directed at problem solving in mathematics. For a general review of the psychological research literature on problem solving, Davis (1966) must serve since as far as I know there is no similarly comprehensive, more recent review. I know of no more recent comprehensive review of research in mathematical problem solving than that of Kilpatrick (1969). For suggestions about actual heuristic methods of going about problem solving (and as the basis for many research and intervention experiments in the literature), one cannot neglect Polya's How to Solve It, while Landa (1975) outlines another approach to problem solving heuristics.

Beyond those general reviews and expositions there are thousands of reports of research on problem solving and hundreds more appear each year. In commenting briefly on that enormous literature, we may as well begin in the way that reviews in this field often begin, with this lament:

> Research in human problem solving has a well-earned reputation for being the most chaotic of all identifiable categories of human learning. The outstanding quality which leads to this conclusion is the diversity of experimental procedures called "problem solving" tasks. The tasks found in problem solving literature range from matchstick, bent nail, and jigsaw puzzles through anagram problems, and even include some mental testing devices such as analogy problems and number-series problems. It is almost definitional of laboratory problem-solving experiments that virtually any semi-complex learning task which does not clearly fall into a familiar area of learning can safely be called "problem solving." (Davis, 1966, p. 36)

That is, even the definition of problem solving is not clear in this body of laboratory based research. Most of that research settles for some version of defining a problem as any situation where a person wants an answer and does not have immediately available either the answer or practical algorithm for getting the answer. As indicated by Davis, such a definition makes nearly anything fair game for research under the problem solving rubric, with the result that Current Index to Journals in Education (CIJE) indexes about 200 articles or theses each year under "problem solving" and Psychological Abstracts (PA) indexes at least as many, most of which do not overlap the CIJE listings.

Whatever its usefulness elsewhere, most of that literature has to be called irrelevant for our concern with applied problem solving. The level of much of it is indicated by the list of nearly 100 laboratory problem solving tasks in an appendix in Davis (1973). An alphabetical sample will indicate the range: A is for algebra word problems, B is for bent-nail problems, C is for card tricks, E is for embedded figure tests,...M is for matchstick problems,...S is for stick and banana,... W is for water jar problems, and so on. I assure you that there is not much there for "applied problem solving."

In addition to the chaotic nature of the laboratory-based research on problem solving, there are other aspects of that research that make it largely irrelevant to those concerned with improving present day school instruction. (That does not, of course, make it irrelevant to everyone.) The research is seldom done in classroom settings with school age youngsters; indeed, in at least half of the research reports I looked at, the experiments were done with paid or volunteer college students (often psychology students). Also, most of the experiments involved short time spans--a few hours at most. We have already noted the narrowness and artificiality of the experimental tasks. All that makes application to in-school instruction difficult, to say the least.

But for all that, one can pick up from that research some very nice rhetoric and insights about problem solving processes. I often suspect that the experiments seem to have provoked those insightful comments rather than supported them, but that does not lessen their attractiveness. Perhaps that is what keeps us worshiping at that altar.

One should observe that the current fashion is to regard "information processing" models as the probable way out of the wilderness, and the promise of the computer simulations of problem solving processes of Newell, Simon, and others is nearly always commented on in reviews since about 1960. There are interesting features in the information processing models, some of which suggest practical things to attend to in instruction in problem solving. For example, the supposed existence of and apparent limitations on "short term memory store" suggest that failure to teach efficient ways to organize, compress, and conveniently record data or intermediate results could make most people fail at any but very simple work on real problems with real data, just from temporary overloads of information. On the other hand, my sampling of the information processing research leads me to conclude that it has at least two basic weaknesses, both of which have also been commented on by others. The first is that in spite of assertions to the contrary, the computer based simulations seem almost inevitably to come down to algorithmic approaches to what we know for sure often requires heuristic processes. The second is that, again contrary to assertions, the regimes as programmed are often far removed from how reasonable humans would probably proceed--certainly far enough removed that they are questionable as experimental evidence. As for applied problem solving, the information processing research that I have seen simply never deals with real problems with real data. (I would welcome being directed to exceptions.) Whatever its usefulness eventually in conceptualizing some aspects of human problem solving, I don't think that we can look to present or foreseeable information processing models as guides to instruction for applied problem solving.

## Applied Problem Solving and Research in Mathematics Education

I made an extensive sampling of the reports on mathematics education research on problem solving but not a comprehensive survey. In that sampling I found many studies concerned with geometric proof or with standard arithmetic and algebra word problems from textbooks or tests, but essentially nothing concerned with applied problem solving. If my sampling missed interesting or significant research concerned with solving real problems with real data, I will appreciate being directed to it. In the meantime, let us note that others who have surveyed the mathematics education literature have noted a similar lack:

> Much has been done to investigate the learning process, though it is a fact that most of this research has been rather laboratory than classroom oriented. Very little, if anything, is known about how the individual manages to apply what he has learned, though such knowledge would be the key to understanding why most people never succeed in putting their theoretical knowledge to practical use. (Freudenthal, 1968, p. 4)

Even if not much attention has been devoted to applied problems as such, it is still worth asking what research in mathematics education has revealed about other aspects of problem solving. As with the psychology-based research, I found in my sampling of U.S. mathematics education literature interesting single results, nice insights, and, especially lately, some promising clinical and classroom-based research methodologies, but little that would serve as reliable guides to instructional practice. (As I will discuss presently, the Soviet literature seems to me somewhat more promising.) My impressions agree with those of the late Ed Begle on his very useful attempt at a comprehensive survey of the vast body of research results in mathematics education (Begle, 1979). In nine chapters and a summary chapter, each with many subheadings, Begle gives his considered judgment of the net thrust of studies in each of many areas. For the most part what emerges from this work is a gloomy picture of non-results, and nowhere more than in the various sections of his problem-solving chapter. For example, he concludes that 63 studies attempting to relate problem-solving abilities to other cognitive abilities "have been no more successful than attempts to characterize mathematical ability" and that "simplistic efforts to improve... problem solving abilities will not be enough." He is similarly pessimistic with respect to studies involving extraneous data (5 such), problem format (25), problem structure (26), strategies (75), instructional programs (36), and verbal variables (21). (But in the case of the last two categories he feels that further investigations might be especially profitable.) He concludes that "compared to the importance of the topic, the amount of factual information that is available to us is quite small" (Begle, 1979).

As I've noted, my sampling of the U.S. literature confirms the validity of the Freudenthal and Begle conclusions. It would be unfair to characterize mathematics education research in this area as a total loss and, in particular, I believe it to be a better starting place than the psychological research. But a great amount of work focused mainly on solving of textbook problems has yielded very little even on that restricted topic, while work focused on applied problem solving using real problems with real data seems scarcely to have begun.

Recently, reports of work in the USSR on problem solving and other areas of mathematics education have become more available in English. Examples include several works of L.N. Landa published in the U.S. and a number of translations from Izaak Wirszup's Survey of East European Mathematical Literature, based at the University of Chicago. That Russian work is clearly quite different from most work done in the United States and since it is still relatively unfamiliar here I will take a few pages to summarize some of it.

Krutetskii (1976) tells us of a number of "teaching experiments" that sort out how mathematically gifted elementary and middle school children go about solving typical algebra word problems. Various aspects of such problem solving are neatly conceptualized and the problems are cleverly grouped into sets to elicit information about each such aspect. (The problems are, however, the traditional ones of school books; essentially none are real problems with real data.) One of his main conclusions is that mathematically gifted children go at solving algebra word problems in qualitatively different ways from ordinary children. That is interesting but perhaps not quite relevant to a concern that most people become confident solvers of applied problems. However, Krutetskii's methodology is certainly suggestive for research in applied problem solving. For example, we could adapt his use of specific problem sets to illuminate specific problem solving processes. Also, we could very profitably do much of our research via "teaching experiments" as used by Krutetskii and other Soviet researchers. In these experiments one does not simply present problems in written tests on a sink-or-swim basis, but presents them in teaching-observation settings where help and encouragement is offered, with careful recording of the process. (Of course, Brownell and others have used similar methods.) Thus, for example, a child can say "Well, I'm stuck, because..." and the experimenter, having noted the difficulty (which may be precisely a difficulty that should arise in sorting out real problems) can offer further information and see what the child does with it. With such experiments focused on process rather than merely on correct answers we could map naturally occurring difficulties and development of skills in school situations as well as exploring specific pedagogies and heuristics for increasing children's abilities in applied problem solving. Furthermore, by various researchers using the same well designed sets of real problems with real data in a variety of situations, information might begin to accumulate in a way that does not now happen.

Krutetskii found that some of his "very capable" youngsters could work only in a representational mode and others only in a symbolic mode, each going to considerable lengths to effectively use their preferred mode even when not appropriate. (Most of his gifted children, however, switched modes as appropriate.) The demonstration of such preferences should surely warn us that our own exclusive preoccupation in school work with symbolic presentation of problems to be done using only symbolic methods is probably harmful even to some of our brightest children.

In several works available in English, L.N. Landa puts forth more provocative ideas about problem solving than I found in any other source (Landa, 1974, 1975, 1976). For example, he neatly deals with distinctions between algorithmic and heuristic methods and observes that most problem solving cannot be primarily algorithmic. He observes that youngsters often possess all the knowledge needed to solve given problems, yet cannot do so. Psychologists or teachers say that such children "can't think properly" or give other vague descriptions of the difficulties, but Landa impatiently dismisses these "explanations" as begging the question and suggests that we attack such problems directly by teaching children to "think properly." In remarks reminiscent of the information processing point of view, he asks us to imagine thinking processes as consisting of certain stored knowledge, certain operations, and certain other mechanisms linked up in ways as yet unknown. But instead of attempting to program that for a computer--which becomes excessively detailed and almost inevitably algorithmic--he asks that we apply such a conception to a human machine, where we can work with larger fundamental units and can assume that appropriate operations and transitions can be elicited without concern about the inner details of how that happens. In one example, he applies that sort of conception to teaching a group of children who demonstrably have all the prerequisite knowledge of geometric facts and theorems needed, yet can do only 25% of the proofs their knowledge should equip them to do. His teaching experiment exploiting a specific set of heuristics resulted in 87% performance and, Landa says, qualitative improvement as well (Landa, 1976). This surely is a very intriguing existence proof and suggests that similar specification of heuristic routines should be tried on behalf of applied problem solving. In a refreshing acknowledgement that both algorithm based and heuristic based instruction have their uses, Landa also speaks of a teaching experiment for Russian grammar that works the algorithmic side of the street, with excellent results.

Three of the books in the series Soviet Studies of Mathematics Learning and Teaching (Kilpatrick & Wirszup, 1969-1975) deal with Soviet work with problem solving in school. In Volume 3 there are articles on problem solving in arithmetic, in Volume 4 articles on problem solving in geometry, and in Volume 6 articles on instruction in problem solving. The articles are variously devoted to pedagogic commentary and to reporting results of teaching experiments and other inquiries. While this work generally lacks the controls, statistics and other manifestations of "precision" that characterize U.S. research in mathematics education, I find them a rich source of ideas and "probably true" results. There is far too much in those volumes to summarize here, but one of the nicest features of them is the provision in the editors' prefaces of neat summary descriptions for the articles of each volume, and I recommend to you at least a reading of those prefaces. But even this excellent literature pretty much ignores applied problem solving.

It should be noted that instruction in applied problem solving as a practical concern is not limited to school mathematics, but is also a concern in training of professionals in business management, medicine, nursing, the military, engineering, and so on. Training regimes such as the use of "case studies" have been formulated and extensively used in those fields, and some of them are described in Davis (1973). My sampling of that training literature was sparse and left me merely with two impressions: first, such work is potentially quite relevant to mathematics education and second, scholarly studies of the actual effects of such instruction are scarce. I may be wrong on the latter point and a survey-review should be undertaken that includes a literature outside both psychology and mathematics education. Also, "problem solving" may have been studied in science education in ways that may be closer to applied problem solving than the typical investigation in mathematics education and I have not surveyed that literature.

## Some "Obvious" and "Necessary" First Steps

As with other persistent concerns in mathematics education, such a sampling of abstracts of research, of full research reports, of reviews of research, and of expository articles and books as I have described leads to this question: Why with all that attention and wise exposition have we made so little headway with instruction focused even on traditional problem solving, let alone applied problem solving? Part of the answer no doubt lies in the general lack of serious attention to problem solving in school work. But I now feel that much of the problem lies in the failure of the university based research, development and teacher training establishments to address the problem at an appropriately "practical" level. That is, there are some obviously necessary first steps that must be taken in order to make headway on instruction with an applied problem solving emphasis. But instead of attending to those basic things, university based efforts are likely to wander off into more "sophisticated" (and hence perhaps more "respectable") studies, most of them fundamentally based in library or laboratory, with little understanding of the school settings for which they seek to prescribe. Even the more "practical" of mathematics education research and curriculum development is likely to labor mightily at such things as making instructional routines out of heuristic schemes such as those of Polya, with little realization that even Polya's very nice insights are probably at a deeper and more detailed level than is likely to be appropriate in the present state of school instruction.

In defending the level of exploration sought by information processing models, Newell and Simon have used the analogy of work in chemistry done with "certain simple assumptions about the combination of atoms in molecules, ignoring the detail of internal atomic structure" (Newell & Simon, 1972). That may be a useful analogy for us as well, although we may need to frankly admit that in our real understanding of what might be fruitful in problem solving instruction, we are about at the level of earth, air, fire, and water as fundamental units.

However that might be, it would surely be wise in considering how to make applied problem solving a viable school emphasis to begin by concentrating our efforts at whatever level we understand pretty well, and to cease to ignore the obviously necessary first steps that are not attended to at present. Here is a partial list of some of those:

1. We must surely begin to attend to the public's expectation that children will not only learn how to do arithmetic but will also be able to use it in everyday affairs. As noted above, the NAEP results indicate that we do the former fairly successfully for simple arithmetic in isolation, the latter very poorly. The minimum competence usually expected for uses of arithmetic is not a high standard and should surely be achievable, but it is probably best embedded in efforts to achieve a much higher standard.

2. It is surely necessary that children actually attempt to solve problems in order to learn problem solving; whether that is sufficient or not remains to be seen. In particular, if children are to learn to use arithmetic, one would suppose they should have experience with applied problems--real problems with real data. As noted earlier, they simply just do not get this experience in the elementary school grades nor in high school algebra or geometry, though some junior high school textbooks do a better job.

3. We have not even begun to face the issues suggested by an old article in *The Mathematics Teacher* with the title "Word problems or problems with words?" (Manheim, 1961). That is, we know that such problem solving as we do give youngsters is almost exclusively oriented to verbal presentation. We also know that many youngsters have serious reading problems--especially in comprehension of the sort of text in which problems are embedded. Hence our exclusive preoccupation with verbal presentation, verbal response, and symbolic processing make such problems essentially inaccessible to many youngsters. That is, we make certain that the poor (in school skills) get poorer (in problem solving). But I don't believe all problem solving skills are inherently linked to reading skills and we should try for a separation whenever possible. Of course, some presentation

of problems in print is inevitable in practical school instruction, so we should also be sure that specific reading skills are taught that will help children extract the problems from the print.

4. Similarly, we need to face up to the barriers to applied problem solving imposed by computation demands that can't be met by children in the early grades simply because the relevant skills haven't yet been taught or by a significant minority later on because the skills haven't been learned. There are at least three ways to begin to attack these barriers. First, the evidence from the Japanese and Soviets is that much more computation skill can be accomplished much earlier in school than we manage, and those existence proofs indicate we could likewise do better. Second, the now universal availability of inexpensive electronic calculators offers obvious but still largely untested potential for minimizing arithmetic skill as a barrier to applied problem solving. Third, there are many interesting real problems with real data that do not make heavy calculation demands, and many of these can be interesting in the early school years. (See, for example, Bell, 1975).

5. In surveys I have conducted of what youngsters in grades 1-3 can do with numbers, each youngster is asked to tell what each basic operation symbol means and then say whether he/she has ever used that operation outside of school. The answer is essentially always "no," unless one counts homework or helping siblings with arithmetic, and that result is confirmed by other research. In other words, children see no connection between school arithmetic and what they do out of school. We get the same results when we ask algebra students to name some uses they make of mathematics. Until we find ways to help children find links between in-school learning and out-of-school life, we are unlikely to be very effective in applied problem solving instruction.

6. We know that children (and indeed people in general) will often enthusiastically engage a difficult and challenging task in a playful setting or one of special interest to them that they would reject if put to them as required school work. But we also know that our attempts to teach mathematics as well as our rare efforts to teach "heuristics" or "problem solving strategies" could rarely be characterized as "playful" and seldom even as "interesting." There is usually something obvious to be attended to here.

7. We know that by now there are many interesting problem and "project" materials available, with a subset of it including interesting applied problems. For example, for elementary school children there are the Nuffield Primary School Guides, Workjobs (Baratta-Lorton, 1972), Teaching Children Through Natural Mathematics (Dwyer & Elliot, 1970) and a lot more. There are also a number of "new science" programs with many nice applied problem solving opportunities; for example, Elementary Science Study, Science: A Process Approach, Science 5-13, Unified Science and Mathematics in Elementary School, and others. For middle school and high school there are several sets of problem solving workcards (e.g.,

Judd, 1977), Statistics by Example (Mosteller et al., 1973), The Man Made World (ECCP, 1971), Mathematical Uses and Models in Our Everyday World (Bell, 1972), Algebra Through Applications (Usiskin, 1976), and much else. We must surely ask why it is that in perhaps 99% of schools, no such materials will be in use in any classroom, and a high percentage of teachers will know nothing even of their existence.

As the recital above should make clear, there are some fairly obvious things to attend to with respect to any sort of problem solving as a school emphasis, let alone applied problem solving. I suggest we get about the business of attending to some of these obvious things without for the present worrying much about deep psychological processes or optimal heuristics in problem solving. Of course, attending to obvious practical concerns need not lead to neglect of more theoretical studies, but I want to urge as strongly as possible that we at least work on these obvious problems, whatever else is or is not done.

## Some Research and Development Initiatives on Behalf of Applied Problem Solving

With the pessimistic analysis given above of our accomplishments to date in teaching applied problem solving it may be useful to close with some suggestions about research and development efforts that might improve this state of affairs. These suggestions will readily be seen as related to the discussion just concluded of "obvious" things we have not yet attended to.

To keep these things in perspective, we must realize that better instruction with respect to applied problem solving is only one of the things that needs to be attended to in mathematics education. I would argue that, on the whole, secondary school mathematics education has things well enough in hand that this additional emphasis could be our major improvement effort for a while. This is especially so since the main trouble spots still at the secondary level (such as "general mathematics" and statistics) would benefit most from such an emphasis. But in elementary school mathematics instruction there are other important things that are at least as neglected as applied problem solving. For example, we need to attend to the emptiness of the primary school mathematics curriculum; to the use of more concrete work in the early years; to finding a proper role for calculators and related devices for concept building or drill and practice or problem solving. But this paper is about what we need to do for applied problem solving, so let us deal only with that.*

First, we need to conceptualize better what might be meant by "applied problem solving" in the lives of most people in an age of cheap calculators, and we need to clarify what we would hope to accomplish if we were doing a really good job with instruction in this area. Many current statements of goals for mathematics and science education speak to such questions (NCSM, 1978; Bell, 1974). But these general statements need to be translated into practical guidelines for instructional planning, with illustrative examples that will communicate to teachers, teachers of teachers, researchers, and authors of curriculum materials. I believe considerable progress could be made on such translation and clarification by a core group of ten people or so in conferences or summer working groups, operating much in the style of the SMSG planning and writing groups of the 1960s.

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*Applied problem solving and some of these other things can, of course, be worked on jointly.

Next, we should recognize that the tendency of mathematics education (and education generally) to periodically "reinvent the wheel" is at least very inefficient. It is safe to say that much of what would be useful to a new and perhaps more practical emphasis on applied problem solving already exists. Hence a number of what might be called "status studies" should be undertaken to maximize the use of existing materials and knowledge. For example, we need to assess the extent to which curriculum materials produced during the numerous development projects of the past twenty years (usually with NSF or other public funding) can be exploited in teaching the practical uses of mathematics. A number of examples of materials from that curriculum reform literature come to mind that are in the applied problem solving spirit. Many of these are very nice indeed, but most either never made it into general school use or they have passed out of general use. It should be said that relatively few things useful to applied problem solving come from the mathematics reform efforts as such; they come instead mainly from projects on behalf of science, social studies, statistics, or engineering. But that does not prevent them being useful in mathematics education. Here, the right few people with access to NSF archives, time and money for visits to former project directors, a WATS telephone line, and a secretary could perhaps unearth and make more available quite a lot of valuable material. They might even be able to assess similar resources known to exist in at least Canada, Holland, West Germany, and Great Britain.

In the same spirit of using rather than ignoring whatever progress has been made, we should seek out existing exceptions to our general neglect of classroom instruction in applied problem solving. That is, in the several millions of school classrooms worldwide, there are almost sure to be classrooms where an excellent job is being done in teaching applied problem solving. It surely can't hurt to find as many of these as we can and study them for clues to ways of duplicating their success in other places.

Still in the spirit of making better use of whatever progress has been made, we should try to winnow out from the research and pedagogical literature such clues as there are to more effective practical work in this area. I don't suggest the conventional comprehensive review of the literature but a highly directed search for fragments from here and there conducted by one or more people with the right instincts. That search should certainly include work done outside the United States.

In addition to such exploitation of existing resources, there must also be creation of additional resources for curriculum, instruction, and teacher training. For example, I have already cited several sourcebooks of applications for use in school mathematics. These are valuable but insufficient, and many more school-usable real problems need to be developed using information and real data from a wide variety of fields. I can testify from firsthand experience that such translation of raw material into good problems for school use is difficult, but it is demonstrably possible and must be done.

Even with a rich supply of problems and other instructional material from the status studies and additional sourcebooks, it is difficult to make applied problem solving a viable emphasis in a given school course, as I can again testify from firsthand experience (Bell, 1970). The difficulties spring from at least two sources. First, there is the force of the traditions of the courses themselves, which are already full of "essential content" and have no room for such "extras" as applications and problem solving. A second source of difficulty is the fact that most teachers, by training and experience, have little knowledge of applications of mathematics and statistics.* It is my present belief that these two barriers are best worked on together, by a variety of experiments with working teachers that first aim to make those teachers better informed about applications of mathematics and then puts them to the task of finding ways to incorporate that knowledge in the courses they teach. That suggestion is not mere speculation, for we have demonstrated the fruitfulness of combined exercises for teacher training and curriculum innovation in several small-scale projects at the University of Chicago. On a larger scale, there is every reason to believe that involvement of working teachers in writing teams and in school tryouts during the 1960s curriculum reforms constituted powerful in-service training for the teachers involved, in addition to providing essential input to those curriculum projects. We cannot recapture those bygone days, nor should we try, but I believe that asking teachers to adapt for their own use a rich supply of "almost-classroom-ready" applied problem solving materials (rather than just handing over finished products) would prove to be very effective both for teacher training and for implementation of new emphases.

Even if I am right about the fruitfulness of combined teacher training and adaptation of problem solving materials for practical use, there remains the necessity of more conventional curriculum development. Such development is needed in the first instance for the creation of the rich supply of "almost ready" materials to support the combined training and development exercises just spoken of. Development is also needed of finished products for those teachers who have no access to the training/development exercises.

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*A third category of difficulties may arise from resistance of students and their parents to any departure from tried and true school mathematics content, but we can't know about that until we try some such departures.

There also remains the necessity for a number of experiments in effective teaching of applied problem solving in pre-service teacher training programs. As a minimum we should try to insure that teachers don't begin their careers already incompetent in such an important matter. Many of the same "almost ready" and more finished products spoken of above will also prove useful in teacher training, but room must be made in teacher training programs for their use, and this is likely to raise some new difficulties.

Finally, and very important, a number of "teaching experiments" need to be carried out, first to find the barriers to more successful work with applied problem solving and then to find practical ways of overcoming such barriers. In my opinion such teaching experiments combined with the better resources that would come from the other initiatives listed above offer virtually our only hope of progress in this area. The sort of laboratory experiments that characterize psychological research in problem solving will, I believe, continue to be sterile of results likely to influence practical classroom teaching. Likewise, the curriculum treatment and effects studies that have characterized most mathematics education research have little prospect of yielding results likely to change instruction in applied problem solving. At this writing, however, I see little prospect that researchers in either psychology or mathematics education will undertake such teaching experiments. Hence, I see little prospect of changing our dismal performance to date with respect to teaching youngsters to use the mathematics that we teach them.

## Some Closing Remarks

My pessimistic analysis of where we stand with respect to school instruction in applied problem solving might be summarized like this: Instruction in applied problem solving using real problems with real data essentially does not exist in today's schools. Perhaps because of that we are getting new evidence daily that while people seem to learn quite a lot of arithmetic in schools, they are often unable to make use of that in their everyday or working lives. Psychological research on problem solving is voluminous but almost completely unhelpful in resolving this dilemma.

Mathematics education research is only slightly more helpful. The mathematics curriculum reforms of the 1960s left this problem virtually untouched. The "new science" curriculum materials of the same period produced much that would help increase people's capacity to cope with applied problems, but little of that is presently used in schools. Furthermore, teachers remain untrained and uninformed with respect to genuine applications of mathematics. In addition, they are presently under pressure from an "accountability" and "back-to-basics" movement that asks only for computational skill and not for applied problem solving. Also, other pressures on schools and teachers make it unlikely that they can from their own resources change the present emphases in school instruction. University scholars in mathematics education and other fields that may be able to round up additional resources for such a task are at present simply too remote from schools to make the required collaboration likely.

On the other hand, we are at least much more aware now than formerly of the kind of trouble we are in. In particular, there seems to be an increased awareness of the need to teach not only the doing of mathematics but the using of it. That has led to the production of new curriculum and instruction resources related to applications of mathematics which, coupled with existing but unused science materials, give a more solid foundation than we have ever had for improvements in this area. The surprising and rapid dissemination of inexpensive calculators and small computers has suddenly finessed a number of formidable barriers to including applied problem solving in school courses.

If my analysis is accurate, I find it difficult to be optimistic about the prospects for making mathematics truly more relevant in the lives of people. If there is hope, I submit that it lies in mathematics educators reestablishing close links with classrooms and with teachers through a variety of status studies, through combined teacher training and curriculum development efforts, and through a variety of teaching experiments that search out the barriers to improvement and then work on overcoming them. None of that seems likely, but it has happened as recently as the late 1950s that deficient and deteriorating instruction in mathematics has been taken in hand and substantial improvements achieved. The things that worked then would probably not be the best strategies now, but perhaps we can again muster the energy and resources to find ways to substantial improvement in making people competent and flexible in using mathematics as well as in doing mathematics.

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# Problems, Applications, Interest, and Motivation

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## An Unlikely Story

Ms. Hope E. Ternal had taught three years. She was dissatisfied with the ways she had taught area and volume. Temporary results were fine. Her eighth graders could learn three or four formulas and use them on the Friday test. But forgetting was rapid, and after a few weeks or months they would use area formulas interchangeably with volume formulas.

Inspiration struck as Hope read the morning paper. She saw "Precipitation 5.8 millimeters." Why not create a context for the study of area and volume? Why not use rainfall--covering areas of the city with volumes of water?

She had tried finding "applications" that held "natural" interest for most of the girls and boys in her previous classes. It wasn't easy to find one application of general interest. This would be different. She would try to arouse interest among most members of a class. She would take time to build a context. The study of area and volume would grow out of this.

Hope talked with her friend the general science teacher. Together they visited the weather bureau. They asked questions and collected printed material. They borrowed a rain gauge. They discussed teaching strategies. Here's what Hope finally did in class:

Friday: Most of the hour Friday was used for a test on decimals. One question asked: If March rainfall was 6.17 cm, April rainfall 7.92 cm, and rainfall for March, April, and May 17.37 cm, how much rainfall came in May?

After the last test paper was collected, ten minutes were left. Hope placed a transparency on the overhead. In large capital letters it said:

PRECIPITATION 5.8 mm

It took almost a minute for talkers to stop talking; for readers to cease to read; for dreamers to quit dreaming. You could almost hear people puzzling. Why did she show us that? What does that have to do with math? Then came the assignment:

1. Write five questions that this newspaper item raises in your mind.
2. Name three of four mathematical ideas that a person trying to answer these questions would need.

Hope resisted the temptation to give examples of acceptable answers. But she permitted some talking; and she spoke with several students asking them questions like: Do you think you could measure 5.8 mm of rain? and Did it rain at your house yesterday?

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Monday: Hope began class by asking a question of her own: How much water is in 5.8 mm of rain? Would it be a liter? Several liters? Or a few milliliters? She suggested that the class might "make it rain" by turning on the water sprinkler on the lawn outside the classroom, collecting and measuring the water. She produced several "fancy" water collectors she had brought from home: a square cake pan, a rectangular pan, an oval meat-roasting pan, a round pie pan, and an angel-food cake pan with a hole in the middle. She asked where the pans should be placed to collect water from the sprinkler, and one of the students suggested that the sprinkler probably threw less water the farther you were from it. The class decided from this suggestion that all the pans should be an equal distance from the sprinkler, so she divided them in teams to place each pan exactly 175 cm from the sprinkler head. They then turned on the water and returned to the classroom. While the sprinkler ran, Hope asked the students to volunteer other questions they had written down for homework. One student wrote questions on the chalkboard, labeling each with the asker's name. Students at their seats copied the questions in their notes. Several questions asked how the weather bureau measured rain. Hope asked if anyone had a rain gauge at their house. Two students raised their hands, and Hope asked them to bring the rain gauges to class tomorrow.

She then called the class's attention to the pans outside under the sprinkler, and suggested that the amount of water in each pan could be measured in two ways--the depth with a ruler, and the volume by carefully pouring the water into a graduated beaker. The teams had just enough time to do this before the bell rang.

Tuesday: Hope began the class by asking each team to put its measurements of depth and volume on the blackboard. The class agreed that the depth of water was just about the same for all the pans except the angel-food cake pan. However, the volume of water measured in the graduated beakers varied considerably. One student suggested that this was because of the different sizes of the pans. They listed the pans by size according to height, but this list did not match the order of the different amounts of water. Several students disgustedly remarked that the size that was important was the "bigness of the bottom of the pan" and not the height. Hope asked how this "bigness" might be measured. Several people remembered a formula for the square and rectangular pans, but the pie pan, oval roaster, and angel-food cake pan stumped them all. Finally, Hope produced some graph paper, suggested that the pan bottoms could be traced on the paper and the number of squares enclosed could be counted. The teams did this, but not without much arguing about how to count parts of a square for the pie pan, oval roaster, and angel-food cake pan. Towards the end of the class period, Hope had students exchange data and then gave the homework assignment: Use your pocket calculators to find a relationship between height, bottom (area) and volume of water that is the same for every pan. Try all the combinations of addition, subtraction, multiplication, and division that you can think of.

Wednesday: No one in the class was able to find a relationship that was the same for every pan. Some students thought the whole thing was quite disgusting, but three students thought they had found a relationship for all the pans except the angel-food cake pan. Two students found that the height of the water times the area of the bottom of the pans came pretty close to the volume of the water (if the decimal points were ignored). The third student found that if one divided the volume of water in each pan by the height of water in the pan you got a number that was close to the number for the bottom area of the pan. There then ensued a discussion as to whether these were two different relationships or the same relationships. They finally decided that the relationships were indeed the same, and that they could be summarized as

Volume = height x area

Several students thought that there ought to be a way to calculate area rather than counting. For the square and rectangular pans it was rather obvious that the area was simply the length times the width, but the pie pan and oval roaster didn't work out so neatly, since they couldn't even decide what was the length and what was the width in those cases. Hope had anticipated this and had dittoed several different circles on more graph paper. She passed out these dittoed sheets and asked students to use their pocket calculators at home that evening to look for a relationship between the number of squares within the circles and the number of squares across the circle at its widest point.

Thursday: This day was spent exploring the mysteries of pi. Hope was proud that she had even worked some history into the lesson. However, she was somewhat apprehensive when she learned that Mr. Former down the hall had covered area and volume in just two days. Towards the end of the period someone mentioned about the odd angel-food cake pan that didn't fit any of the patterns. Some students thought that this was because of the hole in the middle, but someone else suggested that it was because the pan had sloping sides. Hope decided that it was time to get out the rain gauge she had borrowed from the weather bureau along with the rain gauges that the two students had brought in last Tuesday. The three rain gauges looked like this:

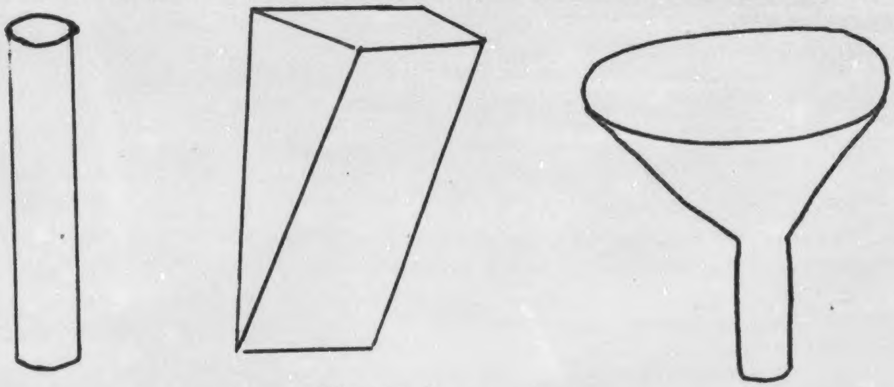

Figure 1

Why did two of the rain gauges have sloping sides? How would one calibrate them to measure millimeters of rain? Would this help in measuring tenths of millimeters? Someone suggested putting all of them underneath the lawn sprinkler again. Someone else suggested that while they were at it they might make a "map" of the different amounts of water that fell at different distances from the sprinkler head. Other students wanted to figure out how many liters of water would fall on the parking lot at the nearby K-Mart if it rained 5 mm overnight. Someone else said, "Why not do it for all of the city?" Then someone else shouted, "You can't do that because not everyplace gets the same amount of rain at the same time." Hope thought about the monthly precipitation maps that she had seen in the newspaper. Perhaps she could use those next week. And she suddenly remembered that her next-door neighbor was a city engineer who once mentioned that he was working on new storm sewer plans.

Hope sighed. There was absolutely no way she could get this class to catch up to Mr. Former's. Perhaps if she just kept the whole thing a secret from the other math teachers....

This is a paper about problems, applications, interest, and motivation. To what extent are these elements present in the unlikely story sketched above? Does the sequence described involve a problem (or problems)? As all too many people (both researchers and teachers) understand problem-solving today, the answer must be "no". Observing 5.8 mm of rain does not present a "problem". A "problem" looks like this:

> It rained 1.6 centimeters in Greensboro on Monday. It also rained Wednesday. Greensboro received 4.0 centimeters of rain Monday and Wednesday. How much rain did Greensboro receive Wednesday? (Bolster et al., 1975, p. 101)

What distinguishes the series of activities tried by Ms. Hope E. Ternal from the textbook problem above is the idea of context. The rainfall activities stem from general physical situations. Indeed, finding and formulating reasonable problems are the first steps in the activity sequence we have described. Problem-formulation is a skill that has been ignored in the present mathematics curriculum. Yet it is probably the key step in studying mathematical applications (as opposed to mathematical problems).

Differences between mathematical applications and mathematical problems are primarily differences in the degree of emphasis placed upon context. Obviously, emphasizing context adds an extra level of complexity to the problem-solving process. Contexts may be unfamiliar, difficult to understand, or just down-right confusing. As the Cambridge Conference Report suggests, they may be internal or external. Internal contexts apply mathematical techniques to new areas (settings) of mathematics. In the (internal) context of quadratic equations, most Algebra I students are expected to apply the technique of factoring polynomials learned in earlier chapters.

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This is an internal application of mathematics to mathematics. In general, the people who are today clamoring to add applications to the mathematics curriculum have completedly failed to recognize the entire category of internal applications. The distinction between internal and external applications is a logical one. However, we need to determine whether or not it is also a psychological distinction. Is there any observable difference in the problem-solving strategies of children when they are faced with internal or external applications?

## Familiar and Unfamiliar Contexts

There are many other questions about the contexts of applications that should be investigated. Perhaps the most obvious of these is the question of context familiarity to students. We quote again from the Cambridge Conference Report:

> To be meaningful, external applications require a knowledge of another discipline. The added concepts required for any applications compound the difficulty of understanding the mathematical material at hand, unless the student is already acquainted with them. It is useless and can be harmful to introduce applications whose context the student does not understand. At best it is then a relabeling of the student's mathematical entities. At worst it both confuses the new mathematical context and causes misunderstanding of the other subject matter. (1963, p. 21)

The effect of unfamiliar settings on problem-solving has received little attention since the 1930's. The benchmark study in this area was conducted by Brownell and Stretch (1931). They presented fifth graders with four arithmetic problem types in four forms each, the forms ranging from familiar to unfamiliar. Unfamiliar problems dealt with nonsense words such as brets, graks, shulahs, bimlechs, toros, pushnas, and chukets! Drawing upon criticisms of previous studies by Washburne and Osborne (1926), Hydle and Clapp (1927), and Washburne and Morphett (1928), Brownell and Stretch carefully controlled a series of variables. The same numbers and operations were used in each form of a problem type. The number of words used in each form was kept the same, and sentence structure was made parallel. The order of presentation of problem form and problem type was rotated among subjects to control possible effects of practice. Teachers' ratings were used to corroborate the four degrees of familiarity for each problem form. Problems were scored for correct or incorrect choice of operation and for correct or incorrect computation.

Results of the study showed that unfamiliar settings did adversely affect problem solution. Correct solutions ranged from a 64% average for the most familiar form to a 51% average for the most unfamiliar form. The source of error was incorrect choice of operation. Errors in calculation remained essentially the same across all problem forms.

However, further analysis revealed that only two of the four problem types contributed to the familiar-unfamiliar differences. One of the problem types that did not contribute to the differences involved averaging. The word "average" was used in each of the four forms of the problem. It seems reasonable that "average" is such a strong clue as to the mathematical operations required that variations of context have little additional effect. Indeed, a "strength-of-clue" analysis of common words (or mathematical terms) in different contexts may be a new profitable way to approach the unfamiliar-context questions.

The other problem type which did not contribute to the differences was worked correctly by more children than any other, and seems to have been an easier problem than the other three. Oddly enough, the problem involving averaging was the most difficult of the four problem types. This led Brownell and Stretch to conclude that familiarity of context is not an important factor for problems that are inordinately easy or inordinately difficult. This conclusion seems plausible; however Brownell and Stretch did not statistically test for interaction of form with difficulty and a modern replication of their experiment seems necessary before drawing conclusions.

Brownell and Stretch also noted other possible interactions. Differences were noted according to whether children encountered the most unfamiliar form first or last. As might be expected, the more that familiar forms were encountered before the unfamiliar form, the better the performance on the unfamiliar form. More time was required by children to solve problems in unfamiliar form; thus the amount of available time interacted with the familiar-unfamiliar factor. Brownell and Stretch conclude: "Provided that a problem lies within a certain range of difficulty, that the particular group of number relationships has not been met too frequently, that the time for solution is somewhat limited, and that the child is not overly proficient in choosing operations and in computing accurately, under all these conditions an unfamiliar setting may lead to the ommission or incorrect choice of operations." (1931, p. 75)

As noted, Brownell and Stretch measured familiarity of context by teacher ratings. Lyda (1947) selected thirty of the most realistic problems from fifth, sixth, and seventh grade textbooks, then constructed an "experience checklist" so that students could rate directly their experiences (familiarity) with various problem contexts. Children were then given the problems to work. Although statistical tests were not employed, the results seemed to confirm the results of Brownell and Stretch. In most cases (although not all) familiar context problems were worked more correctly than unfamiliar context problems.

These studies leave little doubt that familiarity of context is an important variable in problem-solving. However, it appears to strongly interact with other variables, and there is no research that explores this interaction with modern statistical methods. We believe that this is an important area for future research. In particular, we need to look at the strength of clues in different contexts and in connections (or lack of connections) between familiarity and reality. Star Wars suggests that the unreal can be very familiar. The hands-on activities of Ms. Hope E. Ternal are in part designed to make the unfamiliar reality more familiar. Is this logical distinction of psychological importance in problem-solving?

## Interests and Contexts

Does the interest of the child in particular contexts affect problem-solving success? Holtan (1964) constructed four programmed instruction treatments on inequalities using four different application contexts: automobiles, farming, social utility, and intellectual curiosity (games). Ninth-grade students were given Kuder Preference Records and assigned to treatments according to a preference match (high interest) or mismatch (low interest). Although all four treatments were equally effective, there were significant achievement differences between high interest and low interest groups on each of the treatments. Both immediate achievement test scores and retention test scores (three weeks later) favored groups whose preference (interest) had been matched with context.

Travers (1965) presented ninth-graders with pairs of similar problems set in different contexts. He allowed them to choose which problem of the pair to work according to their preference. He found that their problem preference was related to interests as expressed on the Kuder Preference Record. However, total problem-solving success was no greater for the situations most preferred by the groups than for the second or least preferred situations.

Cohen (1976) attempted to predict problem-solving success of eighth graders on outdoor, computational, and scientific problems according to interests measured by the Kuder General Interest Survey. He found no significant predictions using multiple linear regression analyses.

It appears from these studies that the effect of interest in applications contexts is important in instructional situations. Although Holtan's instructional treatment was brief (two days), neither Travers nor Cohen gave any instruction in problem-solving at all. The importance of interesting contexts may depend upon the degree to which we view applications as something to be taught. If an application is presented merely as a puzzle, the importance of interesting contexts may be slight.

However, pilot-study work by Metwali at Ohio State leads us to believe that global measures of interest should be interpreted with extreme caution. She asked eighth graders for preferences about problem contexts, allowing them to choose not only an area of interest but an action within that interest as well. For example, students could choose problems about sports, and then within the sport category choose problems about earning wages, buying sports equipment, or breaking sporting records. There was considerable divergence of choice within categories. In general, preferences of the eighth graders tested were more easily categorized by "consumer concerns" than by the usual interest areas. We have not looked at stability of preferences either across socioeconomic levels or time. Nevertheless, we believe that the matter of student interest may be more complex than past research has accounted for.

Then of course there is the question of whether the application context must be inherently interesting or whether it is the teacher that makes the context interesting. Certainly, many of the activities of Ms. Hope E. Ternal seemed designed to build interest in volume measurement--a topic that might seem to have relatively little inherent interest. It may be that interest in context is highly interactive with the teacher's presentation and style of teaching. Is it always possible for a skillful teacher to develop context in such a way that students will become interested in the application?

## Cognitive Levels of Context

Closely related to the way context is developed are the levels at which it is developed. It seems obvious to us that this can be done in ways that are analogous to concrete, iconic, and symbolic cognitive levels. Most applications are presented to children at a symbolic level. Context is usually described for most applications by oral or written words. Data is embedded within the context in raw numerical form. But this level of abstraction is not the only alternative. For example, one can discuss speed-distance-time applications by actually bringing a toy train and stopwatch into the classroom (concrete) or by drawing maps and pictures (iconic) or by simply presenting numerical data. Do these different levels of context presentation make a difference in problem-solving? We do not believe that this question has been seriously addressed as yet.

The many activities of Ms. Hope E. Ternal seemed aimed at presenting context in as concrete a way as possible. The emphasis on class discussion enhances the development of understanding of the context. The time requirements are enormous by usual standards. Are they worth it?

## Motivating Effects of Contexts and Problems

> All mathematical ideas are motivated by applications of some sort: They enable us to solve new problems and to understand situations we did not understand before. ...a concept should always be motivated. Its need in an application is a strong motive. (Cambridge Conference Report, 1963, p. 21)

Surely few of us believe that the purpose of the school mathematics curriculum is to prepare applied mathematicians. The general reason we give for introducing applications into the curriculum is to motivate students in some way or another. But we know essentially nothing about the actual effects of different contexts in motivating (encouraging) students. We operate perilously close to the principle that if we only introduce applications into the curriculum "something good will happen." Indeed, the usual statements about motivating students with problems and applications reveal some of the most shallow and fuzzy thinking going on today.

We have yet to hear anyone publicly acknowledge that problems and applications can turn students off just as easily as they can turn them on. When Bowman (1929) asked children to choose between problems based on (1) adult activities, (2) children's activities, (3) science, (4) puzzles, or (5) computation they overwhelmingly chose computation. This seems to imply that the addition of a context to the first four types of problems made them less appealing to children than minimally-disguised computation. Do problems and applications turn children off? Ask any teacher. Of course they do.

We believe that the key to this dilemma lies in the success rate of children. All of us are motivated to some degree by mathematical problems. For some of us, it is the primary reason we are now mathematics educators. However, we like problems because we are successful in solving problems. People who are not successful at tasks learn to avoid those tasks whenever possible. The child who experiences the thrill of cracking a mathematical puzzle is motivated to try another one. The child who is presented a series of problems or applications which he cannot solve (or which must be solved for him by the teacher or other members of the class) is repelled from further mathematics study. Using problems and applications to motivate the study of mathematics is one of the most dangerous teaching techniques ever proposed. We may win with a few children. But current practice seems to indicate that we lose many many more.

This is an area in urgent need of research and study. At what points do children become discouraged within the problem-solving process? What factors encourage persistence? Do group problem-solving techniques reduce the negative effects of failure?

Ms. Hope E. Ternal tackles a very broad applications context. Does this allow her to individualize so that children may be presented with pieces of the problem that are appropriate to their skills and abilities? Is this the secret behind the successful introduction of applications?

## Problem Formulation

If we are beginning to ask questions about the processes children use in solving problems, we have not begun to realize that there are similar questions about how children formulate problems.

> The teacher of mathematics may recognize the values inherent in improving students' ability to formulate their problems, foreseeing the nature of appropriate solutions, and yet wonder how the teaching of a technical subject such as arithmetic, algebra, or geometry, is to make any significant contribution. This is due in large measure to the fact that at present few mathematics courses provide adequate opportunities for students to practice the analysis of problem

> situations in other than a restricted sense. Most of the "problems" presented are so simplified and idealized that all that remains to be done is to recognize what operations will lead to the answer explicitly called for, and then to perform these operations--a routine task to be completed by a prescribed set of often relatively meaningless steps. Rarely does the student begin with a more comprehensive situation and go through the experience of simplifying and idealizing the problem for himself, so formulating it that he can work upon it and arrive at a solution which he himself conceives to be appropriate. Under these conditions he has no opportunity to realize the many assumptions and restrictions that have to be made in order to formulate and solve even the simplest problems capable of mathematical treatment. (Committee for the Function of Mathematics in General Education, 1940, p. 76)

It is now almost forty years later, and the situation is hardly better. With the possible exception of the USMES materials, popular advocates of applications in the curriculum still ignore the importance of problem formulation. If we are not making progress in this area, it may be time for researchers to suggest why this is the case. In particular, we would argue that the concept of problem-solving as a research area be broadened rather than narrowed. It is one thing to narrow a particular problem for research design purposes. This does not mean that we have to think narrowly about problem-solving as a researchable area, however. By broadening our horizons from problems to applications we may indeed encounter new and significant research questions. We have tried to suggest some of these. Can you suggest others?

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# Teaching Problem Solving in College Mathematics: The Elements of a Theory and a Report on the Teaching of General Mathematical Problem-Solving Skills[*]

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Can students be taught general strategies that truly enhance their abilities to solve mathematical problems? Or are the "heuristics" described by Polya and others merely a summary description of the actions of accomplished problem solvers, and are they essentially without value as prescriptions for problem solving? While many mathematicians are convinced that they employ heuristics, there is little evidence to show that general problem-solving skills can be taught. With some faith in these general strategies, I offered a course based on their applications to mathematics majors at the University of California, Berkeley. This article begins with an overview, first presenting the rationale for heuristics and then balancing that with some practical concerns arguing against their effectiveness in the teaching of problem solving. A means of circumventing these arguments is offered, accompanied by a description of the course I used to do it. Then--what we can and cannot expect students to assimilate is discussed--the power of the heuristics they can learn to use, and the obstacles that prevent them from learning to employ others effectively.

## I. Problem Solving in Perspective: Theory and Practice

George Polya's How To Solve It was published in 1945. That and his subsequent work laid the foundations for the study of general strategies for problem solving in mathematics, focusing on the broad strategies he called "heuristics." Definitions vary, but the following is within the mainstream and compatible with Polya's usage.

> Definition: A heuristic is a general suggestion or strategy, independent of any topic or subject matter, which helps problem solvers approach, understand, and/or efficiently marshal their resources in solving problems.

Examples of heuristics are "draw a diagram if at all possible," "try to establish subgoals," and "exploit analogous problems"; a more complete list is given in section III. In brief, a rationale justifying research into and the teaching of heuristics, would be as follows:

1. Through the course of his career, any particular problem solver develops his own personal, idiosyncratic style and methods of problem

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[*] A similar report, entitled "Can Heuristics Be Taught?", will be published in the Proceedings of the First Amherst Conference on Cognitive Process Instruction, September 1978

solving. Developing a systematic use of these strategies is often a slow and painful process, taking years to mature fully, if it does at all.

2. There is a surprising degree of homogeneity in the way that accomplished problem solvers attack problems. That is, in spite of personal idiosyncracies, there are global similarities in the behavior and methods of "experts" when they solve problems.

3. By observation, beginning perhaps with the introspections of talented problem solvers and later incorporating the systematic observation techniques of artificial intelligence, one might be able to extract the essence of these global approaches. Thus one might distill a global problem-solving strategy, which accurately describes the principles followed (to some degree) by accomplished problem solvers.

4. The distillate extracted in (3) can serve as a guide to the problem-solving process. Students instructed according to this plan could short-circuit the long and arduous process of arriving at (some or or all of) these general principles by themselves.

Most mathematicians who have seen Polya's work are willing to accede to the first three points in the rationale. Certainly my personal experience was compatible with the rationale, up to that point. My problem-solving behavior when I emerged from graduate school was substantially different from that when I was a college freshman; and even a first-year graduate student. To quote Polya's "traditional mathematics professor," "a method is a device which you use twice." If it succeeds twice, you remember using it successfully, and you think of using it when confronted with another problem, it becomes a strategy. Over a period of time some of these strategies remain (hopefully the more useful ones!) and others pass into oblivion. And a personal, idiosyncratic, and more or less stable approach to problem solving evolves.

But the degree of idiosyncracy can be misleading, as one can easily demonstrate with any problem which is accessible to freshmen but sufficiently unusual that it has to be approached, by both students and colleagues, from "scratch". Ask both colleagues and students individually to solve the problem out loud, and observe the process of solution. In all likelihood the "experts" will engage in some form of systematic exploration designed to "get at the heart of the problem" or "see what makes it tick" (and may indeed say something to that effect). The approaches of the novices will, in comparison, seem quite unstructured--even when the students succeed in solving the problems.

Polya recognized this. His work in describing the general problem-solving strategies employed by mathematicians is excellent. I first saw How To Solve It after completing a dissertation in pure mathematics, and was simply amazed at Polya's accuracy. Page after page I nodded my head in agreement with his words, muttering "yes, I do that!" as I read. My response was similar to most mathematicians', and it is safe to say that few mathematicians would seriously dispute claims (1) through (3) above. Number (4), however, is another matter.

Much to my surprise, I discovered that few of the people responsible for training students in mathematical problem solving at the college level actually use Polya's work in any substantive way when giving instruction in problem solving. A colleague who has very successfully coached his university's team for competition in the nationwide W.L. Putnam Mathematics Competition told me that his students did not find Polya's works useful. They enjoyed the books a great deal, but *they neither seemed to solve problems more effectively, nor perceived themselves as having a greater array of useful techniques for solving problems, than before they had read them*. The faculty member who coached the team that won the Putnam Competition that year told me much the same thing.

Those who are entrusted with training students to solve problems have generally followed this pragmatic but successful rule: *ONE LEARNS TO SOLVE PROBLEMS SUCCESSFULLY BY SOLVING A LARGE NUMBER OF PROBLEMS*. In practice, the formats of their problem-solving courses are remarkably consistent. The students are given a set of problems (usually culled from collections of mathematical problems, or from prior examinations) to try to solve for the next class meeting. When the group next meets, solutions (if any) are presented, various approaches to the problems are discussed, and often comments about the application of the particular techniques to similar problems are made. We might describe this approach as an attempt to *accelerate* the process described above in (1) for each student by means of a "concentrated dose" of problem solving accompanied by direct feedback.

For at least some students (generally the most talented ones), this method of learning problem solving is highly successful. Yet this approach has a number of drawbacks. Chief among them is that it is still *up to each student* to make his own personal synthesis of the material that he has seen: to ingest it, place it into context, organize it, and have it accessible for retrieval when appropriate. The product of many hours work and much thought can be irretrievably lost if the network of connections tying those ideas to others is weak. A student may deal at length with a particular problem, and understand it completely at that time. Yet this does not guarantee that the "lesson" to be learned from it, or even merely the solution to it, will be retained in any way by the student. Lacking (or more precisely, being unable to access) the appropriate connections, the problem solver may later find himself staring at the same problem in total frustration, knowing that he has solved it before but is now unable to recall even the general method of approach.

Surely this has happened to the reader more than once in his problem-solving career. The most dramatic evidence I have regarding this kind of phenomenon was given to me by the first colleague mentioned above. It happened that one of the problems that appeared on the Putnam examination had been assigned in his problem-solving seminar before the examination was given. The three members of the university team, and the three alternates, had all seen the problem solved in detail. None of the six were able to solve the problem on the examination.

The instructor who wishes to give a course in problem solving is caught between two extremes. On the one hand he has available an atractive theoretical structure which seems to describe the way "experts" go about solving problems but which has not been shown effective as a prescription for problem solving. On the other hand there is a pragmatic process which does indeed accelerate the growth of problem-solving ability in some students. But that approach is inefficient, seems to be suitable only for a minority of the "better" students, and lacks any sense of theoretical coherence.

The way out of the dilemma, of course, is to attempt a synthesis of the two: a judicious selection of problems presented within the context of an overall problem-solving strategy. There are some obvious, and some not so obvious reasons that attempts to teach problem solving via heuristics have not been terribly successful to date. I shall discuss these in section II. In particular, I shall argue that instruction in heuristics *alone* will almost always prove to be insufficient: students need to be given, and trained in, an efficient means for selecting the *appropriate* strategies for problem solving and, in general, for budgeting and allocating their problem-solving resources wisely. The means for doing this, which I call a *managerial strategy*, will be described in some detail in section III. The framework, consisting of a managerial strategy combined with instruction in individual heuristics, provided the foundation for a course in problem solving I offered at the University of California at Berkeley in the fall of 1976. The course provided clear evidence that students *can* be taught to employ certain heuristics effectively. We will discuss these "successes" in section IV. They are balanced in section V, where I describe what students *cannot* be expected to pick up, and why. Section VI provides a discussion of both some pragmatic and theoretical concerns for those interested in teaching problem solving, and an idea of some directions for future work.

## II. The Major Obstacles

In order for a student to succeed in solving a particular problem through the use of an heuristic strategy, at least three things must happen:

A. The student must have a "general understanding" of what it means to apply the heuristic.

B. The student must have a sufficient grasp of the subject matter at hand that he can apply the heuristic correctly.

C. The student must think to apply the heuristic!

Now (A) and (B) are without question substantive points and worthy of attention. The bulk of instruction on heuristics until now has focused on (A); the subject-matter competence described in (B) is an obvious sine qua non for the specific application of any heuristic. (C), an apparent triviality, is far from that; in fact, insufficient attention to providing students with a means for (C) may account for the failure of many attempts to teach problem solving via heuristics. We will develop this argument after some elaboration of the first two points.

## A. "Understanding" the Heuristic

It is easy to underestimate the degree of sophistication required to understand and use even the simpler heuristics. As an illustration of this, we will consider one heuristic and a series of problems to which it can be applied. First, the strategy:

(*) "Exemplify the problem by considering various special cases. This may suggest the direction of, or perhaps the plausibility of, a solution."

Now consider its application in each of these problems:

1. Determine a formula in closed form for the series

$$\sum_{i=1}^{n} \frac{1}{(i)\ (i+1)}$$

2. Let P(x) and Q(x) be polynomials whose coefficients are the same but in "backwards" order:

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \text{ and}$$

$$Q(x) = a_n + a_{n-1}x + a_{n-2}x^2 + \ldots + a_0x^n.$$

What is the relationship between the roots of P(x) and those of Q(x)? Prove your answer.

3. Let the real numbers $a_0$ and $a_1$ be given. Define the sequence $[{}^{a}{}_{n}]$ by $a_n = \frac{1}{2}(a_{n-2} + {}^{a}n-1)$ for each $n > 2$. Prove $\lim(a_n)$ exists, and determine its value. $n = \infty\ \infty$.

4. Two squares, each "s" on a side, are placed such that the corner of one square lies on the center of the other. Describe, in terms of (s), the range of possible areas representing the intersections of the two squares.

5. Of all triangles with fixed perimeter P, determine the triangle which has the largest area.

Problem 1 is most often encountered in courses on infinite series, where the clever use of partial fractions reveals the sum to be a "telescoping series." This is entirely unnecessary, however; computation of the partial sums for $n = 1, 2, 3, 4, \ldots$, yields an obvious pattern which can be trivially verified by induction. In problem 2, one employs the heuristic somewhat differently. In the linear case the relation between the roots of $P(x) = a_0 + a_1x$ and $Q(x) = a_1 + a_0x$ is clear. This is obscure when $P(x) = a_0 + a_1x + a_2x^2$ and $Q(x) = a_2 + a_1x + a_0x^2$, however. A choice of conveniently factorable polynomials such as $P(x) = x^2 + 3x + 2$ and $Q(x) = 1 + 3x + 2x^2$ makes things much more transparent, and allows one to suspect the answer. (Proving it is another matter!)

In problem 3 the computations for $a_n$ rapidly become complex. By setting $a_0 = 0$ and $a_1 = 1$, it is possible to compute $a_n$ with ease, and (especially if one draws a picture!) to generalize back to the original problem. In problem 4, one should try a variety of positions where the area is easily calculated; this suggests the desired conclusion. And in problem 5, fixing the perimeter at some convenient value and then calculating the areas for various triangles--including extreme cases, isoceles right triangles, and the equilateral triangle--strongly suggests the result which the problem solver can proceed to verify analytically.

For a relatively inexperienced problem-solver, deriving these five distinct types of actions from the twenty-one words in (*) is by no means trivial. Most often the statement of an heuristic is quite broad and contains few clues as to how one actually goes about using it. The heuristic is not, *in itself*, nearly precise enough to allow for unambiguous interpretation. Rather it is (for the "expert") a *label* attached to a closely related family of specific strategies. "Using the heuristic" in a particular instance means sorting through the family of specific strategies and selecting the one appropriate to the problem.

The more nebulous the statement of the heuristic, or the more difficult it is to apply, the worse the difficulties mentioned above become. Consider the following heuristic, taken from *How To Solve It*.

> (**) If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem?

The range of types of related problems one can construct is immense, but even this is only the beginning. In order to apply (**) successfully, the student must be able to

i) determine an appropriately more accessible related problem,
ii) solve the related problem, and
iii) exploit something from the solution--perhaps the answer, or perhaps the method employed.

The "moral" of this discussion is that learning to use a particular heuristic, even under ideal conditions, is far from simple. What for the "expert" is a label which serves as access to a variety of specific useful techniques is to the naive student a vague and almost useless suggestion. Even illustrating the heuristic "at work" in one or two exemplary cases is insufficient; the student must see it interpreted and applied in a variety of contexts, and then be given training in, and feedback on, his use of it, if we expect him (or her) to use it in any reliable way. In brief, we must be as serious about instruction in heuristics as we are about any other mathematical technique (for example, using the quadratic formula); with any less than that kind and degree of classroom attention, we cannot realistically expect students to learn to use heuristic strategies. (For a more detailed description of pedagogic "necessities" and a sample classroom hour, see Schoenfeld, in press, c).

## B. Prerequisite Subject-Matter Competence

Clearly the student must have some grasp of the subject matter at hand in order to solve a problem. But equally important in the application of heuristics, the student must also have some sort of perspective that enables him to transcend a local level of analysis and sort out essential from inessential details. With respect to the heuristic (**) described above, the student's ability to perform (i) and (iii) may hinge critically on possession of this perspective. One example taken from elementary physics will suffice here. A student had difficulty with the following problem.

> In figure 1, a block weighing 12 lbs. moves on a smooth frictionless plane inclined at 30°, connected by a light flexible cord passing over a small frictionless pulley to a second hanging block weighing 8 lbs. What is the acceleration of the system?

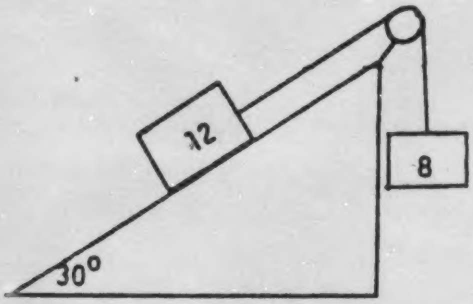

Figure 1

The student was absolutely baffled when asked to create and explore a "more accessible related problem." From the teacher's point of view, some aspects of the problem were critical and others merely technical complications; he could construct and examine a problem which retained "the essential elements" of the given example. But the student found the problem as a whole so confusing that the heuristic was valueless. This kind of "cognitive overload" is extreme, but reflective of the difficulties one can encounter. Much more subtle factors can have similar debilitating effects, as we shall see in section V.

## C. You Can't Solve It if You Don't Think of It

What is obvious is not necessarily insignificant, although it may be all too easily ignored. Such is the case with the truism above. It may reflect the most significant reason that attempts to teach problem solving via heuristics have failed in the past.

> The fact that a student has learned to employ a series of individual heuristics does not in any way guarantee that he will solve a heterogeneous collection of problems effectively, even when he clearly demonstrates the necessary subject-matter competence discussed in (B). The student needs an efficient means of sorting through the heuristics at his disposal and determining within a reasonable amount of time which heuristic is appropriate for approaching the problem--a means of assessing and allocating his resources which we will call a managerial strategy. Lacking a competent managerial strategy, the student may squander his heuristic resources so badly that he loses the benefits he might obtain from them.

In perhaps the most suitable form of argument for this paper, let me plead the case for this statement by looking at a more accessible related problem. Consider the problem of teaching students in a first-year calculus class to perform indefinite integrals with some degree of effectiveness. In this analogy the various techniques of integration--substitutions, parts, partial fractions, etc.--play the roles of the various heuristic strategies in general problem solving.

In (Schoenfeld, in press, a) I indicate that the obstacle preventing students from learning to integrate effectively is not that they have difficulty learning to apply each of the particular techniques. Most students can learn to apply each of the standard techniques--when they know it is the technique they are supposed to be using--with at least some facility. Rather the obstacle is that, when faced with a problem out of context (say on a test or in a set of miscellaneous exercises), students often find it quite difficult to select the technique appropriate for employing on the problem. (If you doubt this, I suggest you give two versions of the same examination on integration to a class. Let the problems on the two tests be identical, but let each problem on the second test be accompanied by a suggestion of the most appropriate method of solution. The differences in student performance will be dramatic.)

In the experiment described in (Schoenfeld, Note 1) a group of students' provided with a managerial strategy for approaching integrals easily outperformed a control group, in spite of spending less average study time per student for the exam. The differences were achieved because the students in the experimental group could marshal their resources more effectively than the others, rather than because of additional competence in applying the particular techniques--they had received no extra practice in that.

When we turn from indefinite integration to general mathematical problem solving, we see that the role of a managerial strategy is perforce more significant. Even when (as is the case with integration) the problem-solving techniques are nearly algorithmic in nature and the domain of problem solving is sufficiently small that we might legitimately expect students to develop reasonably effective managerial strategies on their own, they do not; and providing them with replacement strategies yields significant results. In the realm of general mathematical problem solving the techniques are heuristic. They are often subtle and difficult to apply. The problem solver can be lured down long and torturous mathematical "blind alleys" by pursuing the "wrong" heuristics, so the need for guideposts is greater. And the domain of problem solving is immense. We cannot expect students to develop anything near efficient managerial strategies on their own--especially when they are still learning to employ the heuristics themselves. If they are to "see the forest for the trees", it is incumbent upon us to provide them with the perspective.

The point to keep in mind here is that the managerial strategy must be prescriptive rather than descriptive--and in sufficient detail that students can learn to execute it reliably. When we read the strategy given on pp. xvi-xvii of How to Solve It, the much deeper treatment of strategies in Mathematical Discovery, or an elaboration of problem-solving techniques such as the one provided in Wickelgren's How to Solve Problems, we must remember that the important question is not "does this reflect the problem-solving behavior of experts" but "Is the presentation of the strategy sufficiently explicit that we can expect the student to learn to employ it reliably?"

## III. The Prescribed Remedy

We may take (A) through (C) on pages 40 and 41 as a rough description of what behavior we would like students to exhibit, over a broad spectrum of problems, when they have completed a course on problem solving. One way to guarantee (or at least raise the probability) that the students have (B) is to require junior standing as a mathematics major for admission to the course. With this we can assume that the students have seen a reasonable amount of mathematics, and we have a fair amount of latitude in choosing our examples.[*] It is the instructor's responsibility to provide (A) and (C): that is, training in the heuristics and a strategy for applying them.

Polya and Wickelgren together provide access to just about all the heuristics one could reasonable desire. Thus the major theoretical problem facing me before offering the course was to design a reasonably efficient managerial strategy. Under the assumption that the students can be taught to employ certain heuristics with some reliability, I could think of the students as "information processors" with more or less well-defined attributes. In terms of Artificial Intelligence (AI), the managerial strategy I sought to design would be an "executive program" for the information processor--and the systematic observation/distillation/modeling cycle typical of AI[**] ultimately yielded a strategy which was detailed enough to be implementable and to serve as the foundation for the course. The fully elaborated strategy, even in its nascent form, was quite complex and would be overwhelming to students (it would scare them all of class the first day!); I provided them with an outline and told them that we would elaborate upon it during the course of the quarter. The outline of the strategy, in the form of a flow chart, is given in figure 2 below.

Of course the diagram in figure 2 was too sparse to be useful to the students when I gave it to them. It served rather as an indication of things to come and a frame of reference. During the quarter the strategy was "unfolded" one step at a time. At the appropriate time, each individual box in the flow chart was elaborated upon in detail. Part of that elaboration consisted of a listing of the heuristic strategies most likely to be of use in that stage of problem solving (see figure 3), and instruction in the individual heuristics. The class hour(s) devoted to the strategy "examine special cases" would, for example, be devoted to having the students work the problems (1) through (5) I gave as examples in section 2. The class format was also an important part of the course. Since the major emphasis in this approach to problem solving is on process rather than product, classroom dynamics has to reflect this. Class meetings were true "discussion sections" where each problem was examined in detail. The students were encouraged to develop the solutions on their own, with contributions from myself kept to a bare minimum. After we succeeded in solving a problem (as many ways as possible!) I

---

[*] By the time they have reached their junior year in mathematics the students are certainly "ready" for the techniques. If the students in my course were at all representative, we cannot assume that they have developed either the heuristics or managerial strategies on their own. Some arguments for and against easing the requirements for admission to the class will be discussed in Section VI.

[**] See (Newell & Simon, 1972) for the most elaborate description and (Schoenfeld, in press, a) for a short description of the rationale and process behind the development of the managerial strategy for integration.

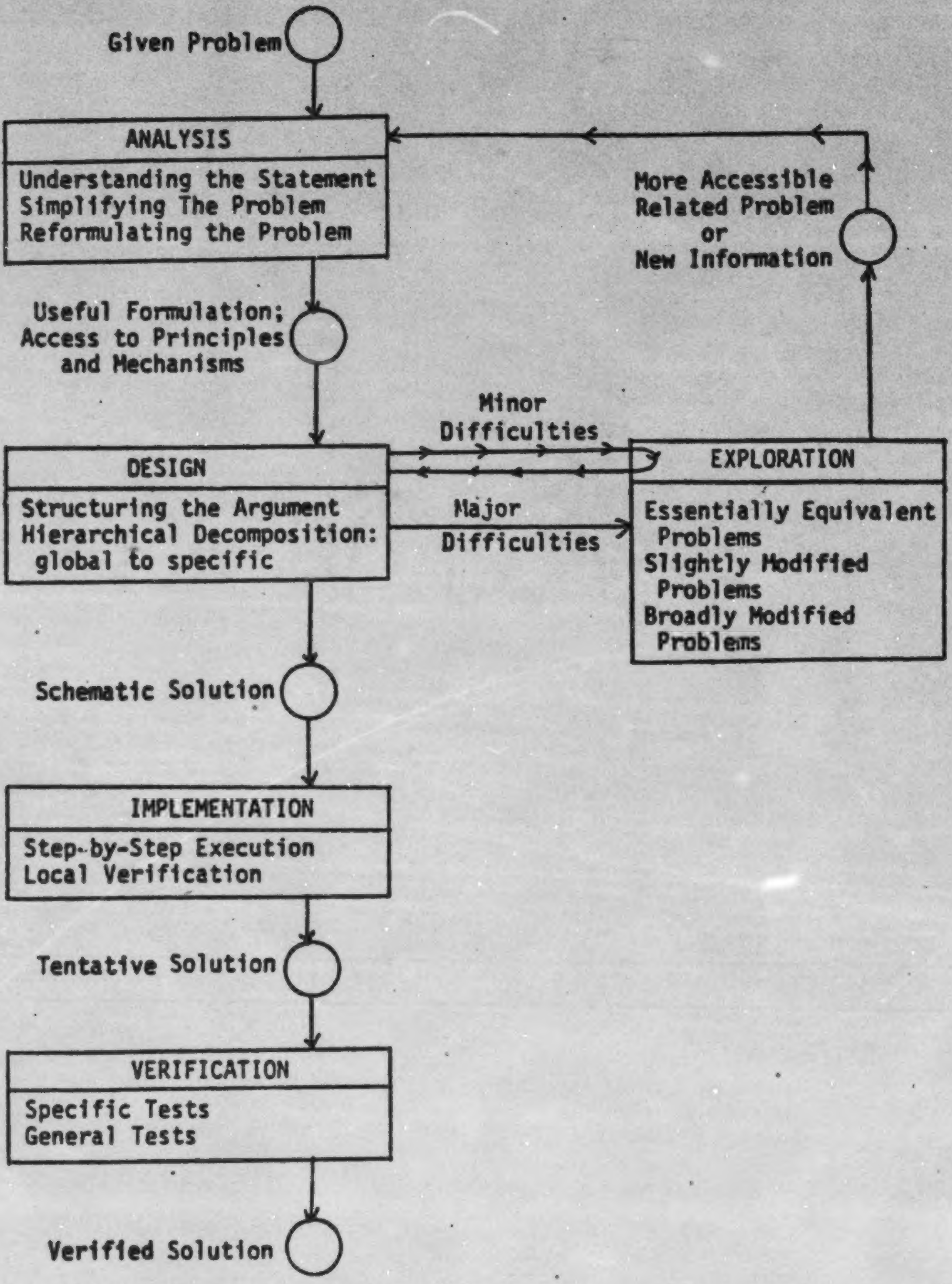

Figure 2. Schematic outline of the problem-solving strategy

would sum up, indicating the way the heuristic was applied in that case and how it might be applied in related problems. Whenever possible we worked within the context of the global strategy, to reinforce that as well as the application of the particular heuristic(s). What follows is a brief, and necessarily incomplete, "tour" of the strategy. The reader should remember as he proceeds through it that the strategy is meant to serve as a guide, not as a straight jacket. Each sentence should be read as if it were followed by the phrase "with all other factors being equal." Some circumstances which tend to make certain factors not equal will be discussed after we have "toured" the strategy.

The first stage of the problem-solving process is ANALYSIS. It begins, of course, with the reading of the problem. It may be said to be successfully completed when the problem solver has a useful formulation of the problem in a convenient representation, a sense of orientation and a mathematical context for the problem, and access to some mechanisms for a close examination of the workings of the problem. Colloquially speaking, after ANALYSIS one has a "feel" for the problem and a sense of what "makes it tick." The teacher should stress the importance of the often underplayed acts of categorizing and establishing a context for a problem. The accurate classification of a problem often accesses immediately a set of procedures appropriate for dealing with it: for example, recognizing that a particular problem is a "maximization" problem tells one that it will be appropriate to find an analytic representation of the quantity of interest, and to use the calculus to find its maximum. Another often underplayed aspect of ANALYSIS is the importance of selecting an appropriate form of representation for a problem. Familiar examples from the literature are the fact that "number scrabble" is a difficult game unless one knows that it is isomorphic to "tic-tac-toe" (see Newell & Simon, 1972), or that "connection lists" can be quite difficult to handle unless one has a visual representation for them (see Hayes, 1966.) We can make the point dramatically to students, however, simply by asking them to multiply the Roman numerals MMCDLXVII and MMMDCCCLXXXIV.

We had in figure 3 a listing of the particular heuristic strategies which (most often, with all other factors being equal) come into play during ANALYSIS. In section II we saw how complex even one of those strategies, "examining special cases" can be. Thus I can only hope to give the "flavor" of ANALYSIS here. Perhaps the best way is to work an example through that stage of the strategy. One final comment before I do: the heuristics listed within any stage of the strategy are given as a set, and not as an ordered set. Any of them may be appropriate to bring to bear at any time during ANALYSIS (or, with a smaller probability, elsewhere).

Sample Problem: Find the area of the largest triangle which can be inscribed in a circle of radius R.

Sample ANALYSIS: As one reads the problem, there is an orientation

Figure 3. A List of Our Major Heuristics

### ANALYSIS

1) DRAW A DIAGRAM if at all possible.
2) EXAMINE SPECIAL CASES.
    a) Choose special values to exemplify the problem and get a "feel" for it.
    b) Examine limiting cases to explore the range of possibilities.
    c) Set any integer parameters equal to 1, 2, 3, ..., in sequence, and look for an inductive pattern.
3) TRY TO SIMPLIFY THE PROBLEM BY
    a) exploiting symmetry, or
    b) "Without Loss of Generality" arguments (including scaling).

### EXPLORATION

1) CONSIDER ESSENTIALLY EQUIVALENT PROBLEMS:
    a) Replacing conditions by equivalent ones
    b) Re-combining the elements of the problem in different ways.
    c) Introduce auxiliary elements
    d) Re-formulate the problem by
        i) change of perspective or notation
        ii) considering argument by contradiction or contrapositive
        iii) assuming you have a solution, and determining its properties
2) CONSIDER SLIGHTLY MODIFIED PROBLEMS:
    a) Choose subgoals (obtain partial fulfillment of the conditions)
    b) Relax a condition and then try to re-impose it.
    c) Decompose the domain of the problem and work on it case by case.
3) CONSIDER BROADLY MODIFIED PROBLEMS:
    a) Consruct an analogous problem with fewer variables.
    b) Hold all but one variable fixed to determine that variable's impact.
    c) Try to exploit any related problems which have similar
        i) form
        ii) "givens"
        iii) conclusions.

Remember: when dealing with easier related problems, you should try to exploit both the RESULT and the METHOD OF SOLUTION on the given problem.

### VERIFYING YOUR SOLUTION

1) DOES YOUR SOLUTION PASS THESE SPECIFIC TESTS:
    a) Does it use all the pertinent data?
    b) Does it conform to reasonable estimates or predictions?
    c) Does it withstand tests of symmetry, dimension analysis, or scaling?
2) DOES IT PASS THESE GENERAL TESTS?
    a) Can it be obtained differently?
    b) Can it be substantiated by special cases?
    c) Can it be reduced to known results?
    d) Can it be used to generate something you know?

and categorization, probably involving the assumption "this will involve calculus." The problem solver should draw a diagram, and possibly begin searching for an analytic representation. For the sake of simplicity, he might decide to look at the unit circle (scaling), and note (without loss of generality) that one may assume the base of the triangle to be horizontal. If one examines a few special cases--and draws a few more diagrams--he may come to realize that, for any particular horizontal base, the triangle with the greatest height (and thus the largest area) is isoceles. At this point, the problem has been reduced to the following: find the base of the isoceles triangle in the unit circle which yields the largest area. The problem now is more or less of a "standard" one variable maximization problem, and with the appropriate choice of analytic representation, has a "ready-made" plan which will dispatch it as a routine matter.

As the flow chart in figure 2 indicates, one proceeds from ANALYSIS to DESIGN. At the simplest level (which obtains for straightforward or routine problem solving) DESIGN consists "merely" of the intelligent ordering and structuring of an argument. The problem solver should have an overview of the solution process; he should be able to say, at any particular point in the process, what he (or she) is doing, why he is doing it, and how that action relates to the rest of the solution. He should proceed through (all but the most routine) solutions hierarchically, taking care to avoid being immersed in intricate calculations pertaining to one part of a solution if global aspects of another phase of problem solution remain unresolved. (We have all suffered the discomfort of solving a difficult equation, only to discover that it didn't have to be solved in the first place!)

For more complex problems, however, DESIGN takes on more global and significant dimensions. It is different from the other phases of problem solving, in a sense pervading them all. Design is a "master control," monitoring the whole of the problem-solving process, and (as best it can with the information it has) allocating problem-solving resources efficiently. It keeps track of alternatives, so that if the chosen approach to a problem proves more difficult than expected, other approaches to the problem can be considered and (in the light of this difficulty) the most likely to succeed chosen. If there is difficulty in making a straightforward plan, DESIGN sends the problem solver into EXPLORATION. Problems resolved without much difficulty are returned to DESIGN and the elaboration of the problem-solving plan continues. However, if EXPLORATION provides new insights into the problem or the solution process, the "master control" may decide that it is most appropriate to return the problem, with the new information, to ANALYSIS.

From the above discussion it should be clear that DESIGN is the most nebulous and most difficult to prescribe of the stages of the problem-solving strategy. In the classroom it calls for an openness and awareness on the teacher's part of the students' individual problem solving, and for tremendous restraint on the part of the teacher. He (or she) must not say what is "right", but rather help the student to decide what is "right" for him.

EXPLORATION is the heuristic "heart" of the problem-solving strategy, in that it is during the exploratory phase that the majority of heuristics generally come into play. As we see in figure 3, EXPLORATION is divided into three stages. For the most part the suggestions in the first stage are either easier to use or more likely to provide a solution, or provide direct access to a solution, of the original problem than suggestions in the second stage; likewise for the relation between the second and third stages. All other factors being equal, the problem solver would begin EXPLORATION by briefly considering the heuristics in stage 1, and selecting those (if any) which seemed appropriate for trial. When the strategies in stage 1 proved insufficient, he would consider those in stage 2; if need be, when stage 2 has been exhausted the problem solver tries the strategies in stage 3. At any point in the process where substantial progress is made, the problem solver may decide either to return to DESIGN to plan the balance of the solution, or to re-enter ANALYSIS with the belief that the insights gained in EXPLORATION will help re-cast the problem in a way that was not accessible before.

IMPLEMENTATION requires little by way of elaboration, save that (as called for by DESIGN) it should be hierarchical with detailed calculations and such saved for the last stages of solution. VERIFICATION, on the other hand, is deserving of more mention if only because (in practice) it is so often slighted. At a local level, checking over one's solution often allows one to catch silly mistakes. In general, be reviewing the solution process one can often find other ways to solve a problem, see connections to related subject matter, and on occasion, become consciously aware of useful aspects of the solution process which can be incorporated into one's global problem-solving strategy.

Finally, I should return to my earlier comment that the strategy is meant to serve as a guide and not as a straight jacket. The intention of the strategy is to provide students with a useful framework for approaching problems (which is sorely needed); it is not meant to be a rigidly followed algorithm, with the students serving as human automata. Certainly the work described in Schoenfeld (in press,a) indicates that well defined strategies are neither alien nor constricting to students. But more, one must realize that this strategy is meant to be incorporated into the student's own framework and modified accordingly. Further, the phrase "with all other factors being equal" is meant to be far more than a ritual incantation; it serves as a gateway to personal alteration of the strategy. For example, familiarity with a particular problem domain may enable one problem solver to consider in ANALYSIS what another might reach only in EXPLORATION. Also, experience may allow for by-passing some of the strategy; one might start a problem in vector analysis by decomposing the vector, even though "breaking into parts" is formally a stage 2 operation. Similarly, certain strongly "cued" heuristics, or pattern recognition, or "intuition" may lead the problem solver to try a particular strategy "ahead of its time." This should and does happen with expert problem solvers. For example, in a problem solving experiment, five out of five mathematicians individually decided, within one minute, to approach the problem

Let a, b, c, and d be given real numbers between 0 and 1.
Prove that $(1-a)(1-b)(1-c)(1-d) > 1-a-b-c-d$.

by examining the two-variable problem and extrapolating the result. The strong "cue", the presence of too many variables playing similar roles, prompted the action--as it should. At least, it should prompt consideration of the strategy. The "master control" in DESIGN should decide whether or when it is appropriate to use it.

Ultimately, a much more refined version of the strategy might take such "cues" into account. Perhaps the "condition-action" language of production systems will prove an appropriate vehicle for describing such behavior (see, for example, Larkin's "Hi-Plan" model, in press).

## IV. What Impact CAN We Expect Heuristics to Have?

In brief, the major statements I can make as a consequence of my course about students' abilities to learn problem solving via heuristics are as follows:

A) In a wide variety of circumstances, individual heuristics can have a dramatic effect on students' abilities to solve particular problems. Often the mere mention of a particular heuristic can either crystallize what were jumbled thoughts in the student's mind or point out a new direction of attack on a problem, resulting in a solution to a problem that was previously inaccessible. With proper training, students can learn to apply heuristics to difficult problems in rather sophisticated ways.

B) Students can learn the essential ingredients of a managerial strategy. Most important in this regard, they can develop skills in determining the appropriate heuristics for dealing with a wide variety of problems--an absolutely critical fact, in view of the discussion in Section IIC. This means that we can legitimately hope to have impact on students' problem-solving behavior, outside of the heuristics classroom.

A few caveats are appropriate before we proceed. The class was experimental, with an enrollment of 8. On the one hand, the small enrollment allowed for a detailed monitoring of the students' performance and growth--entirely appropriate at this stage of development of the model and the idea. On the other hand, one should always be wary of extrapolating from a small sample in these unusual circumstances, especially when the students received such individual attention. The students claimed that the amount of work they did was "about average"; and frankly, I feel that any course in problem solving, to be effective, will require a large degree of commitment from the instructor. The numbers that I offer in the sequel are in no way an attempt to argue statistically based on a sample of eight students; they are rather an attempt to add quantitative substance to some qualitative statements.

## A. Individual Heuristics DO Have an Impact

As indicated above, the mere mention of an heuristic can often trigger something that was hitherto inaccessible to the student. One dramatic example of this was provided by a problem we discussed in section IIA, where students were asked to

1. Find an expression in closed form for $$\sum_{i=1}^{n} \left( \frac{1}{(i)\,(i+1)} \right).$$

Of the eight students in the class, two had succeeded in solving this as a homework problem. One remembered seeing it as a telescoping series, and the second remembered having made some sort of algebraic manipulation; after a while he saw the partial fractions decomposition for the denominator. In class the other six students said that they had tried but simply "got nowhere" on the problem. At that point I presented them (for the first time*) with the heuristic:

Examine Special Cases:

a) Choose special values to exemplify the problem and get a "feel" for it.
b) Examine limiting cases to explore the range of possibilities.
c) Set any integer parameters equal to 1, 2, 3, ..., in sequence, and look for an inductive pattern.

The students then worked by themselves for a while. Within four minutes all had seen the pattern, and within ten all had verified it. [This example is one of the most spectacular I've seen for convincing skeptics that heuristics can work. Virtually everyone (except the mathematicians who know the telescoping series) is stumped by it at first, and finds the answer (if not the proof) apparent once the suggestion is made.]

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*One of the difficulties in getting students to employ any strategy is that they are unlikely to employ it unless they are themselves convinced of its value. For that reason I would occasionally give students a problem like the above to work on, and later provide them with the heuristic appropriate for solving it. The dramatic turnaround is impressive, and helps to convince them of the heuristic's utility. This is a useful instruction strategy, but it should not be abused: if students are given too many inaccessible problems they can become frustrated, and the approach counterproductive.

Similar results occurred with the following problem:

2. For what values of (a) does the set of simultaneous equations

$$\left\{\begin{array}{c} x^2-y^2 = 0 \\ (x-a)^2+y^2 = 1 \end{array}\right\}$$

have either 0, 1, 2, 3, or 4 solutions?

This problem is not terribly difficult to crank out by algebraic means, although the "bookkeeping" can get sufficiently complex that students make silly mistakes. The problem is number 173 from the U. S. S. R. Olympiad Problem Book, and the algebraic solution is given on pp. 276-277 there. Only three of my eight students handed in error-free solutions.

Now consider approaching this problem *via* the following heuristic:

DRAW A DIAGRAM if at all possible.

The first equation becomes two straight lines intersecting and at 45° with the origin; the second equation a circle of radius 1 with center at (a,0) (see figure 4). With the graphic interpretation the problem is trivial. Again seven students out of eight solved it completely, all claiming the problem was easier to deal with. (Two had used the heuristic before.)

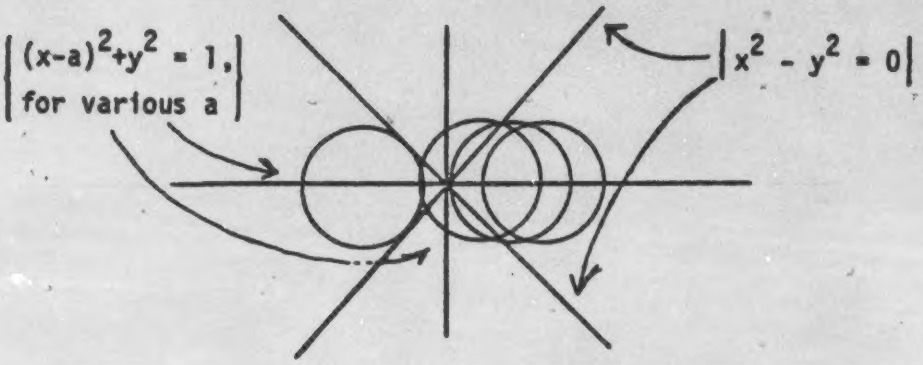


Figure 4

Another case in which an heuristic produced drastic results was in the following problem:

3. Let n be an integer. Prove that if $(2^n - 1)$ is a prime, then n is a prime.

Two students out of eight solved the problem within ten minutes, with the rest of the class making little progress. [Unfortunately the heuristic employed for question 1 yields the sequence of numbers (3, 7, 15, 31, 63, 127, ...), which does not seem especially suggestive; at this point the students were left without any ideas.] The idea introduced to the class at this point was that of trying to make the problem more accessible by re-formulating it, as suggested below.

Reformulate the problem by

i) change of perspective or notation
ii) considering argument by contradiction or contrapositive
iii) assuming you have a solution and determining its properties.

Clearly the appropriate choice among these is (ii). The negative of "prime" is "composite" and the problem becomes 3'. Let a and b be integers greater than 1. Prove that $2^{ab} - 1$ is composite.

The problem was easy for the four of the remaining six students who realized that $2^{ab} - 1 = (2^a)^b - 1$ has a factor of $2^a - 1$.

As a last example of the efficacy of particular heuristics consider the following problem:

4. Show it is impossible to find numbers a, b, c, d, e, A, B, C, D, E, such that

$$x^2+y^2+z^2+r^2+s^2 = (ax+by+cz+dr+ex)(Ax+By+Cz+Dr+Es).$$

The problem is overwhelmingly complex, and the only three students to solve it on their own did so by employing the same heuristic that I later offered to the whole class:

> If a problem containing a number of variables is too complex to analyze, construct an analogous problem with fewer variables and solve that. Then try to exploit either the method, or the result, of the analogous problem.

With that suggestion, another three students succeeded in solving it.

These examples are atypical. One does not frequently encounter problems which are relatively inaccessible to students when presented out of context, and which are completely "unlocked" by the mere mention of an heuristic. (If such problems abounded, the efficacy of heuristics would be unquestioned!) Yet the examples are significant. They indicate that particular heuristics can drastically affect the solution rates on problems, if only because they focus the problem-solvers' attention on methods of attack which they might have used but did not think of using. The impact of heuristics easily transcends this "focusing", however. In the classroom discussion of problem (4), I discovered that only one of the students recalled ever having seen the heuristic explicitly mentioned by any of his teachers before. It is safe to say the problem is too complex for them to solve without recourse (prompted or otherwise) to the strategy. Thus a course in heuristics adds to the students' problem-solving repertoire in addition to giving greater access to some already known techniques. But this claim, and evidence I obtained in the class to indicate that students can successfully use the heuristics described above to solve quite complex problems, while interesting on its own, would be sterile in a practical sense unless we could deal with the difficulties raised in section IIC. It is all well and good that particular heuristics can help students unlock particular problems. But the students should be able, on their own, to determine with some facility which heuristics should be brought to bear on particular problems. At the very least they must have a coherent managerial strategy for approaching problems and calling the heuristics into play. Otherwise the effect of the heuristics could easily be diluted beyond the point of tangible returns. We deal with this in part B.

## B. Students CAN Learn to Choose Appropriate Heuristics

Part of the final examination in my problem-solving course was specifically designed to see if students could, within a short amount of time, select the appropriate means of approaching a variety of problems. The students were given one hour to examine twelve questions, and told to answer the following for each of them:

> For this particular problem, which of the techniques we have studied in class strike you as being most likely to help you
>
> i) understand the problem,
> ii) determine an appropriate means of, and make progress towards, a solution?
>
> How would you approach this problem if you had an hour to work on it, and why (briefly!)? Mention specific heuristics.

After the statement of each question the students were asked if they had seen that (or almost identical) problem before; and if so, how much of the solution to the problem they remembered. Since the course was offered pass/not-pass and they were assured that their answers to this question would in no way affect the grading of their papers, I am fairly confident they answered honestly. (Certainly some of the problems they quoted as being "almost identical" to ones on the examination indicated they were trying to be honest!) A sample of the questions and the responses follows.

1. Let S be any non-empty finite set. We define E (S) to be the number of subsets of S which have an EVEN number of elements, including the null set and possibly S. Determine E (S) in closed form for any finite set S, and prove your answer.

Seven of the eight students indicated that they had not seen the problem before. All of them indicated that they would examine sets of 0, 1, 2, 3, ... elements to see if a pattern emerged; if it did, they would prove it by induction. The one student who had seen a solution outlined a combinatorial argument. The next problem has been used in a variety of contexts. We saw it as an example of "expert" response to "cues" in section III. It was also used as a pretest problem in an experiment where seven students with background similar to those in my course were asked to solve it; only two students of seven considered the "fewer variables" strategy.

2. Let a, b, c, and d be real numbers between 0 and 1. Prove that

$$(1-a)\ (1-b)\ (1-c)\ (1-d) \geq 1-a-b-c-d\ .$$

This problem was new to all eight of the students. Seven of them indicated that they would go about solving it by first examining the one and two variable cases,

$$(1-a) \geq 1-a \qquad \text{(rather trivial!) and}$$

$$(1-a)\ (1-b) \geq 1-a-b$$

The eighth student (not the same as in problem 1) left the problem untouched.

3. Let $C_1$ and $C_2$ be two smooth non-intersecting closed curves in the plane. Prove that the shortest line segment which connects a point of $C_1$ to a point of $C_2$ is perpendicular to both $C_1$ and $C_2$.

None of the students had seen the problem before. (The closest we had come in class to the problem was a discussion of some aspects of the isoperimetric problem.) Not surprisingly six of the students mentioned drawing a diagram, and the other two said something about "trying examples." But in addition four of the students suggested an argument by contradiction, taking the line segment L as given and proving there would be a shorter one if L were not perpendicular to both $C_1$ and $C_2$. Three of the remaining four students noticed the symmetry of the problem statement and observed that it was sufficient to prove L perpendicular to $C_1$; one of these further noted it is sufficient to examine the degenerate case where $C_2$ is a point. (The eighth suggested examining two circles or perhaps two ellipses.)

4. Let f be a function whose domain is ordered pairs of polynomials and whose range is the set of all polynomials:
$f[p(x),q(x)] = r(x)$, where p, q, and r are polynomials.
We define I(x) to be an identity polynomial under f if
$p(x) = f[I(x),p(x)] = f[p(x),I(x)]$ for all polynomials p(x).

Prove the identity polynomial is unique.

The phrasing of the problem was deliberately obfuscatory, for I was curious to see if they would apply (ii) on page 55. Two of the students left the question blank. The other six suggested assuming the existence of two identities and proving them equal, one of them commenting "we do this any time we want to prove uniqueness."

These results are far from conclusive, but they are suggestive. Judging from the quality of their homework papers at the beginning of the quarter, the students' ability to make such determinations improved substantially during the quarter. Certainly one can argue that the fact that students were able to select an appropriate means of approach to a problem in no way guarantees that they would be able to solve it. Selecting an appropriate heuristic is far from a sufficient condition for success, but it may be necessary. If the students did not approach any of the above problems via the means described, they would have had almost no chance of solving them!

## V. What Can We NOT Expect of Heuristics?

The examples in Section IV were chosen to indicate that students can learn to apply a variety of heuristics to a wide range of problems, using a managerial strategy to assist in the selection of the heuristics. Yet clearly there are limits to the problem solving via heuristics that students can learn. In this section we examine some upper bounds on student performance, and reasons for them.

### A. Subtlety in Application May Stymie Students.

The statements of heuristics are often vague, and leave much in the way of interpretation to their users. Consider the heuristic "examine an easier analogous problem with fewer variables" when applied to the following.

1a. Prove that for all real numbers a, b, and c,

$$a^2 + b^2 = c^2 = ab + bc + ba$$

implies that $a = b = c$.
In class I asked what the easier problem to analyze would be. A student answered that by setting $c = 0$, we obtain the simpler problem

1b. Prove that $a^2 + b^2 = ab$ implies $a = b$, and the class agreed.
That choice is inappropriate, of course: the correct statement obtained from setting $c = 0$ in (1a) is

1c. Prove $a^2 + b^2 = ab$ implies $a = b \underline{= 0}$, which is not analogous.
In view of the cyclic nature of the terms on the righthand side of 1a, the correct analogous problem is

1d. Prove $a^2 + b^2 = ab + ba (= 2ab)$ implies $a = b$.
This is easy to solve, and the method used to solve (1d) can be used to solve (1a).

### B. Cognitive Organization and Perspective are Limiting Factors.

The degree to which any particular person can employ an heuristic depends significantly on the way he encodes information and the perspective he brings to the subject matter. As an example, the last lines of the proof we developed in class to problem (1a) read:

(*) Since $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$, we have $a = b$, $b = c$, and $c = a$, as desired.

As an experiment I left the solution to this problem on the blackboard and gave the following problem.

2. Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots b_n\}$ be given sets of real numbers. Determine necessary and sufficient conditions on the $\{a_i\}$ and $\{b_i\}$ such that there are real constants A and B with the property that

$$(**) \quad (a_1x + b_1)^2 + (a_2x + b_2)^2 + \ldots + (a_nx + b_n)^2 = (Ax + B)^2 ,$$

for all values of x.

I told the students that the two problems were related in some way, and that they should seek to exploit the first in trying to solve the second. I gave them fifteen minutes and asked them to work individually. None of the students made any progress towards solving the problem, for none of them saw the structural similarity between (*) and (**).

This is not surprising. Equation (**) is, after all, a morass of symbols none of which are quite comparable to those in (*): the number of terms is different; the quantities are polynomials instead of numbers; there are subscripted variables; and the right-hand side, rather than being zero, is a quadratic polynomial with undetermined coefficients. Nonetheless when I read problem 2, I was immediately reminded of problem 1a, although about an hour had passed since I had selected problem 1a for discussion. In solving problem 1a, I had been impressed by the fact that a great deal of information is contained in an equation where a sum of squares equals zero. When I read problem 2, I saw a sum of squares equal to something. Thus, if I could replace something by zero, I would gain much information. The compact form of encoding "sum of squares equals . . ." enabled me to do this. Anyone lacking such a concise yet powerful means of summarizing the two equations would probably find the structural similarity between them obscured beyond recognition.

Similarly the perspective one brings to a problem (largely a function of one's experience) may subtly determine what he sees in that problem and thus how successful he will be in solving it.

Consider the following problem.

3. Let T be a given triangle of area A. Using a ruler and compass, construct two lines parallel to the base of T such that the three resulting areas (see figure 5) are all equal.

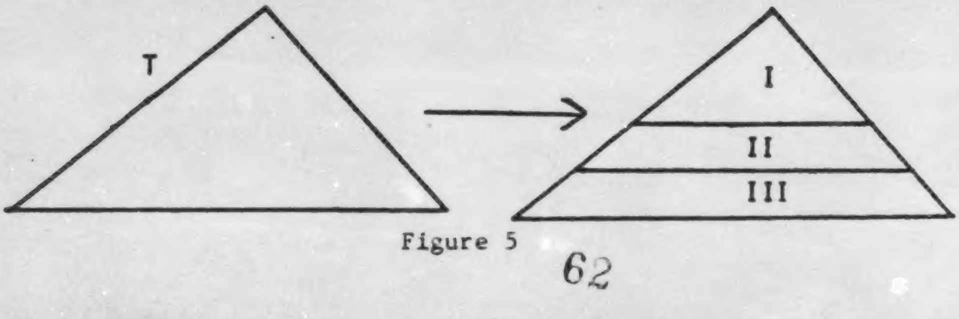


Figure 5

A quite competent graduate student with a background in physics had great difficulty with this problem because he was only able to perceive the problem as illustrated in figure 6. He found it nearly impossible to equate the area of the upper triangle with that of each of the two trapezoids.

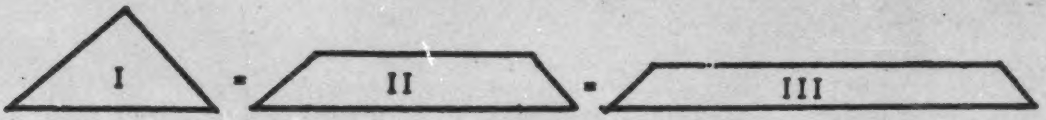


Figure 6

For myself, for a colleague who had taught high school geometry, and for four of the eight students confronted with this problem on my final examination, figure 7 is a reasonable representation of the way our perspective enabled us to see figure 5.

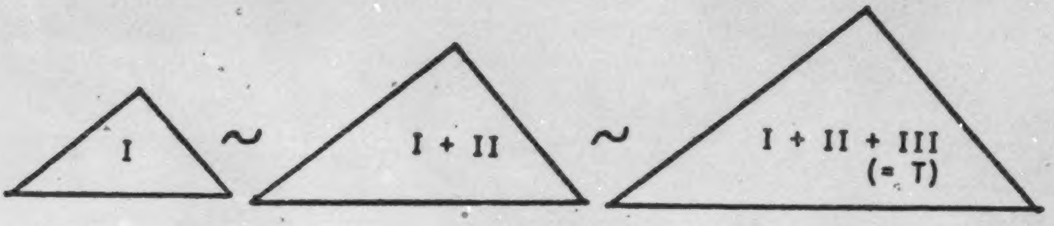


Figure 7

With this perspective we see three similar triangles of areas A/3, 2A/3, and A respectively. Since the areas of similar plane figures are proportional to the squares of their sides, the problem reduces to that of constructing $1/\sqrt{3}$ and $\sqrt{2} / \sqrt{3}$ with ruler and compass. But only those who see the similar triangles in figure 5 will make this observation, and no amount of training in heuristics can compensate for not seeing it.

## C. Induction Yes; Generalization Perhaps; Synthesis No.

The fundamental assumption underlying any attempt to teach heuristics is that, once students have been shown how to apply an heuristic in a variety of circumstances, they will themselves be able to apply it. To a certain degree this held true in my course. By the end of the quarter students were substituting n = 1,2,3,. . ., for integer parameters, even when the parameters were only implicit; they were analyzing the impact of particular variables on problems by holding all but that variable fixed and letting that one vary; and so on. But as soon as the class encountered something that was substantially beyond their range of experience, they ran into trouble. For example, the following was a homework problem.

4. Let N be any integer. Find D(N), the number of distinct integer

divisors of N, including 1 and N.

The table of values for D(N) does not, on first glance, reveal any suggestive pattern.

| N: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| D(N): | 1 | 2 | 2 | 3 | 2 | 4 | 2 | 4 | 3 | 4 | 2 | 6 | 2 | 4 | 4 | 5 | 2... |

At this point the class was stuck. It did not occur to them that the next question to ask in the search for a pattern was: Which values of N give particular values of D(N)? Once I asked them this it became apparent that whenever D(N) was 2, N was prime; that N was a prime square when D(N) was 3; and they were on the way to solving the problem. But this application of "look for a pattern" was beyond the range of their experience and thus not useful to them. (Assumedly this particular method of seeking a pattern may have become incorporated into their interpretation of the heuristic after we solved the problem. But there is no guarantee that they will be at all successful the next time they encounter a problem where an unusual interpretation of the heuristic is called for.)

This difficulty becomes more critical when students encounter a problem whose successful solution depends on the concurrent (synthetic) use of two heuristics.

5. Let P be a polygon drawn in the plane whose vertices are all points with integer coordinates (lattice points). Find a simple formula for the area of P, which depends on the number of lattice points in the interior, and on the boundary of P.

The answer (known as Pick's Theorem) is that the area is $\frac{1}{2}(2I + B - 2)$, where I and B are the number of interior and boundary lattice points of P, respectively. This formula is sufficiently complex that most students are unlikely to hit upon it with a random selection of special cases. The optimal approach to this problem is to combine two heuristics we had studied. One should first fix one variable (say $I = 0$) and then take sequential values $B = 3, 4, 5, 7$; then fix $I = 1$ and repeat the process, and then the same for $I = 2$. At that point the formula should be accessible. But this kind of synthetic approach was not at all apparent to the students in spite of the fact that they had kept variables fixed in the past and had certainly looked for inductive patterns. Likewise, we cannot expect students to combine in synthetic fashion other heuristics they have learned individually.

## VI. Conclusions

The "state of the art" in the teaching of general mathematical problem solving is extremely primitive, notwithstanding the fact that the notion of "modern heuristic" as introduced and substantially developed by Polya has been with us for more than thirty years. The established theory has not progressed to anywhere near the point where it provides an implementable prescriptive model for problem solving; current practice is largely in the hands of isolated individuals, each proceeding on the basis of his personal pragmatic experience. This paper has dealt, often in speculative fashion, with various aspects of a prescriptive theory of problem solving. Pragmatic and theoretical issues have been mixed, for (obviously) they feed off each other. In this concluding section they are separated. In part (A) I will discuss some utilitarian concerns for those who might themselves consider offering a course on problem-solving. In part (B) I present an overview of this theory in its nascent state and indicate some avenues for exploration.

## A. Pragmatic Concerns for the Teacher of Problem Solving

Far more goes into the successful teaching of a course on problem solving than the compilation of the strategies which serve as its theoretical foundation. For example the role of affective considerations, which have barely been touched upon in this paper, is critical. In a domain where the confidence (or lack thereof) of the individual problem solver may decide whether or not he solves any particular problem, we can scarcely afford to pass lightly over such concerns. This is not the place for an extended discussion of the dynamics of teacher-student relationships, however. Let me merely note that the issue is worth serious attention and refer the reader to discussions by Polya (1965, Chap. 14) and Schoenfeld (in press, c).

In preparing for my course, I found two things to be critical. First was the philosophical framework, which I will try to sum up in part (B). Second--and most germane here--is the choice of examples for discussion. If the teacher of problem solving via heuristics takes his responsibility seriously, selecting a collection of valuable and instructive problems for his class may be the hardest task he faces. The purpose of the examples is to illustrate, in a wide variety of contexts, how each general heuristic can prove valuable. Problems which are "unlocked" by a single heuristic or which serve as dramatic examples of an heuristic's utility are not common and, it may be argued, are not reflective of the universe of mathematical problems as a whole. Nonetheless they are absolutely necessary to the process of instruction: they serve both to convince the students of the importance of heuristics and to illustrate their use. Only when the student has mastered the use of heuristics in this sub-universe can we expect him to use them in the large. (The weaning process to more complex problems should take place near the end of the term of instruction.) Some of the problem sources I found most useful are listed in the bibliography. The problems given as examples in this paper are a fair sample of those which meet my criteria. Let me indicate two classes of problems which I feel should be avoided. First, avoid problems for which the given solution depends on what might be called "divine revelation." Consider the following, taken from The U.S.S.R. Olympiad Problem Book (Shklarsky, et al., 1962):

Schoenfeld

1. Prove that $n^2 + 3n + 5$ is never divisible by 121 for any integer n.

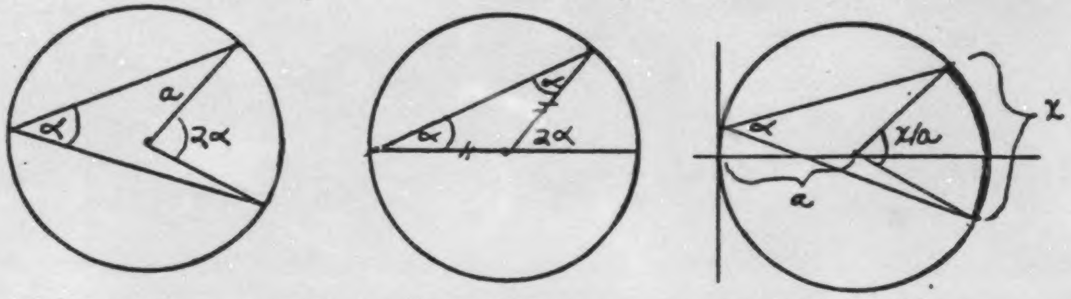
Now consider the way the argument in the book begins:

> "We shall use the identity $n^2 + 3n + 5 = (n+7)\ (n-4) + 33$."

I cannot see how anyone (except the person who constructed the problem) can be expected to arrive at the identity himself, and I see no instructional value in the problem. Second, *avoid problems whose sole justification for inclusion is a subject-matter theme*. The "easy way out" of selecting examples for class is to pick a series of problems from a text (or a collection like the Olympiad book) dealing with a particular subject: divisibility by integers, for example, or cardinalities of sets. There is no doubt that such problems occur frequently in problem-solving competitions and that the student who wishes to do well in such competitions should be familiar with the subject areas. However the techniques in any particular subject area are often domain-specific and may shed no light on general problem-solving skills. A domain-by-domain approach to the teaching of problem-solving is fundamentally at odds with an efficient presentation of heuristics as general strategies, at least at the beginning of instruction. The students can be given problems united by a subject matter theme *after* they have learned to employ a variety of heuristics, as a part of the weaning process mentioned above.

Finally I should comment about the *level* of the problems we can use in a course on problem solving, and the implications of this. Surprisingly, the problems we use as examples need not be as "advanced" as we might expect. Often a problem which is easy within a particular subject matter context can be quite difficult for students when their established patterns of problem solving within that domain have eroded with time. For example, consider the following problem. It is routinely "solved" and often a "required proof" in high school geometry classes.

2. Prove that in any circle, the central angle which subtends a given arc is twice as large as any inscribed angle which subtends the same arc. (figure 8a)



a. "general" case b. "special" case c. parametrization $r = 2a \cos \theta$

Figure 8

Schoenfeld

My intention when I assigned the problem was that the students consider a special case, where one side of the inscribed angle is a diameter of the circle (figure 8b). The general case can be obtained by adding (or subtracting, if the inscribed angle does not include the center of the circle) two special cases. To my surprise, the problem (assigned as homework) caused my students some difficulty. When students encounter a problem out of context, strange things can happen: one person whose problem solving was particularly idiosyncratic solved the problem by parametrizing the circle and performing an arc length integral in polar coordinates. (I pointed out gently that he might have suspected that there would be a simpler proof.)

Thus one need not use only advanced mathematical problems for object lessons. A word of warning in this regard, however: it was good that I gave the problem as homework and not as an in-class example. Students may perceive such elementary problems as insults to their intelligence. (After all, they solved that type of problem routinely four years ago!) Purely as a matter of instructional strategy, we must convince them there is a lesson to be learned from such "sample" problems. This is best done by letting the students discover that the problems are not as trivial as they might have thought.

Since one can learn the "object lessons" of the heurisitics from some problems at a fairly elementary level, it may well be possible to ease the admissions requirements for the course. With a careful selection of problems, the course I offered might be tailored to students who have finished a year's instruction in calculus. I wonder about the benefits of such a course to students with weaker backgrounds, however. It might be more appropriate to offer them a basic course in learning to think and argue in logical, mathematical fashion--something which, justified or not, we assume that upper division mathematics majors bring with them to a problem-solving course. Offering a problem-solving course to people with minimal backgrounds would present difficulties both in finding enough appropriate problems (a major limiting factor!) and in the range of possible exemplification. The lowest level at which these strategies can be presented, with tangible benefit to students, is an empirical question. That it can be done at all, with the results described in sections IV and V, has been established.

## B. Theoretical Issues

For the sake of brevity, I will make the following basic assumptions to provide a context for this discussion.

a. Although there is a great deal of idiosyncracy and individualism in problem-solving behavior, "experts" deomonstrate discernable patterns in the way they approach mathematics problems. They often seem to explore unfamiliar problems through the (conscious or unconscious) use of certain global problem-solving strategies.

b. The essence of these global strategies can be distilled, and described with a large degree of accuracy. Polya has singlehandedly succeeded in capturing the essence of the most important heuristics, with Wickelgren covering whatever ground Polya left open. In addition Polya has treated the affective aspects of teaching problem solving with insight.

c. A substantive number of mathematicians--and scientists in general, to the degree that they perceive the heuristics described by Polya to apply to their disciplines--accept Polya's descriptions of heuristics as valid. They acknowledge having used many of them themselves (most often independently of Polya, only to discover his descriptions later), and would be happy if their students could use them.

d. Notwithstanding (a) through (c), and the fact that there have been numerous attempts to teach general problem-solving skills via heuristics, there is virtually no reliable evidence to indicate that one can substantially enhance students' abilities to solve problems (in any meaningful way) by teaching them heuristics.

Now if we accept these four assumptions, we are forced to draw one of two conclusions:

1. Heuristics such as those described by Polya may well serve as a summary description of "expert" problem-solving behavior. They are, however, doomed to failure as prescriptions or guides to problem solving for non-experts. Unfortunately, each individual must--on his own, over a period of years, and in an undoubtedly inefficient fashion--develop his own personal approach to problem solving;

or

2. Some element is lacking in the established theory of problem solving or in the means that have been used to teach it (or both). If we can extend or modify the theory to incorporate the missing element, we may hope to teach problem solving via heuristics successfully.

I personally am not ready to conclude (1) in despair. In this paper I have discussed, at some length, the reasons that I believe the present theory is incomplete. I have offered a means for supplementing the established theory with the elements that seem to be lacking in it, and given some (rather tentative) evidence to indicate that these ideas not only have some foundation in reality, but can be implemented in a practical course on problem solving. Let me summarize.

Suppose that we can succeed, to a reasonable degree, in training students to apply each of a series of individual heuristics. These may well be substantive additions to the students' problem-solving repertoire. Even so, we cannot expect the students to demonstrate substantially enhanced performance when tested on a range of problems calling for the application of a variety of heuristics. There is another element which is absolutely essential if students are to use these heuristics to advantage: they must be able to select, with some speed, the appropriate means of approach to each problem. If students lack this ability to make an efficient choice of a problem-solving approach to any problem, they may dissipate their heuristic resources beyond the point of tangible returns. It is incumbent on us, therefore, to provide students (as best we can) with an efficient means of selecting the methods they will bring to bear upon problems.

The means I propose is a "managerial strategy", the outline of which was given in section III. The strategy was designed to simulate (within reason) the way that accomplished problem solvers would approach problems that were new to them. It provides for a global allocation of resources in the way that an "executive program" does in Artificial Intelligence programs. It is further buttressed with a series of "cues" which, in certain circumstances, serve to route the problem solver directly to specific approaches for particular types of problems. If the heuristics from the established theory are taught within the context of this managerial strategy, and if the students can learn to allocate their heuristic resources thereby with some efficiency, we may hope to overcome the difficulties mentioned above.

Although the sample of students I had in the problem-solving class was too small to lend real authority to any statement about the implementability of such a managerial strategy, the results I obtained in the class and described in section IV are definitely suggestive. They indicate first that students can learn to employ individual heuristics with some degree of competence. But this is not all. They indicate further that students can indeed learn to select the appropriate means of approach to a wide varity of problems, and that the selection can be done in a short amount of time. In short: there is reason to believe that a managerial strategy--or its equivalent--is the missing element in the established theory of problem solving. With its inclusion we may hope to teach problem solving via heuristics successfully and with demonstrable, replicable impact.

Transforming that hope to documented fact will be a major endeavor, and would be even if the theory were at a far more advanced state of development. The difficulties in obtaining successful and meaningful results in experimental environments are legion. Our experimental subjects are not rats or pigeons but human beings. They are not being trained in the execution of linear S→R chains whose links consist of pressing levers when green lights go on (or the human equivalent, reciting paired associates) but in the most complex and least understood thought processes known to man. They do not enter the experimental area with *tabulae rasae*, and they are not in a virginal state relative to the experimental materials, as one would like for purposes of "ideal" experimental design. Rather--and this is a difficulty one must face if he hopes to perform experiments whose results will translate meaningfully into the real world--each individual brings with him his own personal host of preconceptions, misconceptions, and idiosyncracies which must be circumvented, obliterated, or superseded by the problem-solving strategies which are offered him.

The results described in section IV are suggestive, and indicate that we may well hope to teach effective problem solving via heuristics. Let us not forget, however, what went into the instruction that produced the results. The students met with an interested instructor for three hours per week, receiving (almost) personal attention in the classroom. This was supplemented with a fair amount of homework, for a total of ten weeks. It is safe to assume each student spent in the vicinity of a hundred hours dealing with various aspects of problem solving for the course. Much of that time is probably necessary, although refinements of the theory may shorten the total time needed to produce significant results. Tidy little experiments which will provide incontrovertible evidence of the success of such strategies are simply not in the offering: there remains much to be sorted out before we know enough, or have enough control over instructional variables, to be able to document some of the claims made here. Progress is being made, however. The experiment described in (Schoenfeld, Note 1) shows that, under certain circumstances, we can demonstrate the impact of heuristics. It provides as well a mechanism for further study of heuristic behaviors. The use of protocol analyses for detailed investigations into cognitive process and of production systems for modeling may provide both more information about problem-solving strategies and a convenient language for discussing them. More generally, the emerging field of cognitive science may provide insights into the nature of human thought which we can exploit. In brief, there is indeed reason to believe that we can come to a better understanding of productive human thinking, and that we will be able to use this understanding to the benefit of our students.

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# A Review of Selected Literature in Applied Problem-Solving Research*

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This chapter includes a review of selected literature in several areas of problem solving. No attempt was made here to cover all identifiable areas. The focus is on "routine" applied problems--e.g., the typical textbook verbal problem--rather than nonroutine problems. The first two sections, on problem context and on the physical way in which the problem is presented, supplement the Trimble and Higgins chapter, Chapter 2. The third section centers on the many language factors which have been studied with verbal problems. The final section treats studies which have examined "processes" in problem solving. Each section concludes with some recommendations for research.

## Problem Contexts

As in the Trimble-Higgins paper, the word "context" will be used here to refer to the setting in which the problem arises. For example, the talented Trimble-Higgins teacher, Hope E. Ternal, chose a context involving the volume of dirt in the streets or of garbage collected in the city.

### Interest

One would expect that "interesting" contexts for problems would be related to greater problem-solving success. As Trimble and Higgins point out, neither Travers (1965, 1967) nor Cohen (1976) found such an effect. Travers' subjects, however, solved only their preferred problems in each of 15 pairs. The problems in each pair were matched, but without matching across pairs the success rates observed might be spurious. Cohen chose outdoor, scientific, and computational contexts for his problems but notes that contexts from sports, automechanics, and music might have led to greater predictive power. Pursuing a different line, Blake (1976) studied the influence of context on heuristics used by eleventh graders. Although he did not determine the relative degree of students' interest in his matched real-world context vs. mathematical context problems, he found that the problem contexts did not result in different quantities or types of heuristics used.

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Even though attempts to demonstrate the effectiveness of a match of a student's interests to problem contexts have been largely fruitless to date, one might nonetheless wish to choose problem contexts to meet "average" student interests in some global fashion. DeLong's (1975) survey of pre- and early-adolescents suggests that pets, atheletics, working with the hands, outdoor games, travel, association with peers, spending money, living outdoors, and watching cartoons and comedies might provide problem contexts of interest to many children. Textbooks, of course, vary in the match of their problem contexts with student interests. To compare text contexts with children's interests, Hensell (1956) first reviewed surveys of children's general interests and found these areas to be high ranking: game play, organized sports, movies (note the 1956 date), areas of study, academic subjects, and relations or activities with people. When he looked at two text series, he found that they utilized only 43% and 60% of the 31 general interest areas he isolated.

Another way to attempt to adapt to different interests would be to use sex-appropriate contexts. (The adjective "sex-appropriate" is Leder's, 1976, and is intended to reflect commonly cited interests and leisure activities of boys and of girls.) Leder devised two forms of a problem-solving test, one using activities stereotyped as male (M paper), the other, female (F paper). With 310 Australian tenth-graders she found that "there was a slight tendency for boys to perform better on the M than the F paper ...Girls tended to perform better on the F than the M paper" (p. 123). No means or significance levels were given for these results.

## Familiarity of Contexts

Trimble and Higgins ably discussed the familiarity-unfamiliarity dimension in their chapter. As they note, one discomfiting feature of studies involving familiarity vs. unfamiliarity is, what is "familiar?" Certainly the nonsense words used by Brownell and Stretch would be unfamiliar. On the other hand, Welch (1950) reported a greater interest by fifth and sixth graders in "unreal" problems (with fanciful elements) than in "real" problems (of social significance).

Here is a "matched" pair of problems used by Washburne and Morphett (1928, p. 222), which illustrates what they meant by "familiar" and "unfamiliar":

Unfamiliar A coke plant had to turn out 673 tons of coke for a steel company. It has already produced 129 tons. There are 32 ovens in the plant. How many more tons will each oven have to produce? (53% of the students used the correct algorithms)

Familiar My committee has been asked to dress 363 pencils for the bazaar. We have already dressed 149 pencils. There are 23 girls on the committee. How many more pencils will each girl have to dress? (71% of the students used the correct algorithms)

The performances on the problems in this pair do seem striking. Yet performance on the unfamiliar problem in other such pairs sometimes was as good as that on the familiar one in the pair. (The language section below will touch on the many questions that can be raised about the matching of such problems.)

Brownell and Stretch did not put the matter to rest. White (1934), for example, criticized earlier studies for having had teachers decide what was familiar and unfamiliar. She polled sixth-grade students themselves to determine familiarity and then gave 12 pairs of problems, one within the experience of children, the other not, to 1000 students. She concluded that if a problem was easy, the amount of experience of the child with the situation was not a factor. With other problems, experience was a "highly significant" factor. (Recall that many studies of this vintage did not undertake elaborate statistical analyses. It is perhaps noteworthy then that Post, 1958, did detect a main effect for familiarity of setting, in an analysis of variance.)

The Lyda (1947) and Lyda and Church (1964) studies also involved determining the degree of student experience with the contexts of textbook problems selected for their "realism" and then studying the relationship between the degree of experience and success with the problems. These studies are small-scale and strangely-reported. Yet they do suggest that experience with a situation may be more important for the average and below average student than for the above average student.

Having students write their own problems has often been advocated. Besides providing practice with concepts, perhaps this strategy would result in familiar problems and should play on the motivation of having "one's own" problems. Keil (1964) provided contexts (usually without numbers) for sixth-graders, and contrasted the effects of (a) having the students make up and solve their own problems and of (b) having them solve given problems about the settings. After 16 weekly sessions, she found significant (0.01) effects favoring the group which wrote its own problems. The dissertation does not report means (!) and it is likely that the classroom rather than the student should have been the unit of analysis, but still the results are encouraging. Riedesel (1964) also included the practice of having pupils make up their own problems as part of his successful experimental treatment, but the design did not allow tracing the effect of the problem-writing.

## Conventional vs. Imaginative Contexts

About a half-century ago, there was a sequence of studies on the effect of embellishing the usual story problem with more information about the situation--i.e., an "imaginative" presentation as opposed to a "conventional" one. Conventional problems were marked by simple, compact, direct statements of the problem, whereas imaginary problems described more details of the larger situation, intended to add interest.

Here is a sample of each (from Wheat, 1929, p. 25):

> Conventional  A girl caught 35 of her string of 135 beads when the string broke, and could only find half of the rest. How many beads did she save?
>
> Imaginative  Bertha's daddy bought her a new string of 100 beads. While she was playing tag, the string broke. Poor Bertha! The beads scattered in every direction. She caught 48 of them before they fell, but she found only half of the rest. When she put the beads on another string, she had only .... beads.

Myers (1925) gave six conventional and six imaginative problems to 494 children in grades 5-8 and to 19 normal-school students. The results led him to assert rather definitely the superiority of the imaginative presentation on word problem performance. Wheat (1929), apparently dismayed in part at the size of arithmetic texts which incorporated several imaginative problems, felt that Myers had not proved his case. Hence, Wheat replicated the Myers' study and also tested the conventional vs. imaginative types with other sets of problems. After giving different tests to several hundred students, he concluded that there were no differences in difficulty between the two types, except that conventional problems take less time.

Ten years later Bramhall (1939) reported another study dealing with the conventional vs. imaginative issue. Noting that Myers and Wheat had given only tests, he had 427 sixth-graders work either conventional or imaginative problems three days a week for 10 weeks, with teacher supervision but without a great deal of teacher "influence." Bramhall concluded that the types were equally effective. The data table in the report is for a standardized reasoning-in-arithmetic test, but no data for a 20-problem posttest (10 of each type) are tabulated. Bramhall does mention that the imaginative-problem students performed slightly better (about 5%) than the conventional-problem students on this test. Since imaginative problems take longer to work however, he felt that the better performance was offset by the fact that fewer problems could be worked. It is worth noting that after the 10 weeks, students had gained an average of 8 months on the reasoning-in-arithmetic test.

## Some Recommendations

1. Keil's (1964) study, in which student-generated and student-solved problems apparently led to improved problem-solving with sixth-graders, should be replicated. Perhaps some freedom of choice of contexts could be allowed in an attempt to increase the match of interest and context.

2. Would a concentration on word problems give the same results as in Bramhall's (1939) study: 8 months growth in 2½ months? If indeed problem-solving is as important as many feel, it should be moved from an

"If there's time, ...." status to a central role featuring much problem solving. Until there is a supply of Hope E. Ternals, a stop-gap approach would be to solve lots of problems.

3. Knifong and Holtan (1976) found that reading difficulties may not play as great a role in incorrect problem solutions as is commonly believed. Students do have trouble with problems which include extraneous data (e.g., Post, 1958, James, 1967). Perhaps it is time to revive the conventional-imaginative issue, with extraneous data in each type. Presenting the "full" context with extraneous data is much more life-like, and perhaps the imaginative version better reflects real-life situations.

4. Interest, familiarity, imaginative appeal, extraneous data--what other context variables might play a role in applied problem-solving? Caldwell (1978), for example, found that couching a verbal problem in terms of an unknown number ("If 15 is added to an unknown number, ....") rather than in terms of a concrete referent ("If 15 dolls are added to a collection, ....") resulted in more difficult problems for students in grades four through twelve.

## Problem Formats

In this section, "formats" will refer to what Trimble and Higgins call "cognitive levels of context." For example, a problem might be presented by written or spoken words only, by a picture with minimal use of words, or by appropriate physical props with minimal use of words. A problem about the cost of bicycle accessories could be couched in words, in a picture with price tags, or with actual accessories marked with prices. This section focuses on contrasts between formats rather than within-format variations such as the following. (a) Some texts do list data line by line, perhaps reflecting the influence of the positive results found for such a format in the learning of complicated definitions (Markle, 1975).
(b) Recent studies by Campbell (1976, Note 1) warn of the danger in assuming that student perception of pictorial or concrete formats will be what are intended. Her work with presenting problems by pictures to first graders suggests that sequences of pictures be used before single pictures and that the pupils may interpret postural cues (e.g., legs "moving") somewhat better than cues like clouds of dust or "motion" lines.

### Formats in Tests

When elementary school pupils have been tested, diagrams or pictures have often seemed to result in better performances. For example, Neil (1969) included diagrams to be completed with half the problems given to groups of third graders. She found that performance on the problems with diagrams was statistically superior (0.01). Portis (1973) examined the influence of a physical aid or a pictorial aid (or no aid) when he tested groups of fourth, fifth, and sixth graders on problems involving proportions. Both the physical aid and the pictorial aid groups performed better than the words-only group. On the other hand, O'Flaherty (1971) found no differences between providing a picture (black and white or colored) with a verbal problem, or just presenting the problem orally. Her subjects were 64 fourth and 67 seventh graders, tested individually.

Sherrill (1973) and Webb and Sherrill (1974), in testing tenth graders and pre-service teachers, respectively, found that an accurate drawing accompanying a verbal problem gave better performance than the word statement alone, but that the word statement alone gave better performance than the word statement accompanied by an inaccurate drawing. In contrast, however, Kulm, Lewis, Omari, and Cook (1974) gave 116 ninth graders either a text statement, a student-generated statement, a drawing, a text-statement + drawing, or a student-statement + drawing for a given problem. Only their students with IQ 92-109 performed differently. For them, the text statement treatment was superior, in terms of number of correct or correct method, to any treatment involving a drawing! Kulm et al. opine that perhaps the familiarity of the textbook language and the novelty of the other treatments accounted for this result. Then too, their mode of presentation--one minute via overhead projector, then three minutes to work after the overhead was turned off--was unusual. Only this last study, with its unusual presentation mode, conflicts with this conclusion: Accompanying a verbal problem with a diagram or an accurate drawing in a group-testing situation gives better performance than the verbal presentation alone.

Finley (1962) presented 20 problems to third-graders and to retarded students (mean MA 8 years, 5 months) in three ways: first, with actual money (accompanied by the written question, which was also read); second, with pictures of the money, similarly accompanied; and third, with numerical symbols only, as computation exercises. The third version can scarcely be equated to the first two, so it will not be discussed here. The actual-money format was given first (individually); the picture-format was second, one week later (group). She found that both the third-graders and the retarded students performed better on the picture-format than on the actual-money-format! However, the fixed order of the tests, the individual administration of only the actual-money test, and the students' familiarity with money may account for these differences.

## Formats in Instruction

Nickel (1971) used concrete materials as well as pictures and diagrams in a "multi-experience" treatment which, after six weeks of instruction, produced a performance superior (0.10) to that of a verbal-only treatment on an arithmetic applications test. This difference, however, did not show up when the fourth graders were tested again after three weeks. The design did not allow one to study the influence of the pictures and diagrams alone.

Shoecraft (1972) designed three approaches for teaching the translating of algebra problems: direct translation, the use of drawings (actually diagrams) before translation, and the use of concrete materials before translation. With 12 days of instruction, the 366 seventh graders and 336 ninth graders in no case performed best under the diagrams treatment, on learning, retention, or transfer tests.

Yet, Nelson (1975) found an enhancing effect from instruction which incorporated the presentation of problems via diagrams. Of his 362 sixth graders, the ones who received diagram-presented problems did perform better on such problems, but not significantly better on problems presented only in words.

Boersig (1973), with 98 ninth-grade algebra students, studied the effect on problem solving of videotape demonstrations of, for example, distance-rate-time situations acted out with toy cars, or perimeter problems demonstrated with yarn on a pegboard. In addition, the materials were also present in the classroom, and each student in the experimental group used the material at least three times. Boersig called this use of videotape-plus-materials the enactive mode of representation and used them to supplement programmed materials (using drawings and symbols) on solving coin, mixture, and uniform motion problems. A control group used only the programmed materials. There were no differences on a problem solving posttest with respect to writing appropriate equations, although the experimental group means were higher than the control group means. Experimental group subjects were significantly better (0.10) at writing out the implicit conditions for the problems (e.g., that two distances in a problem were equal or summed to a given distance).

## Formats and Problem Formulation

Trimble and Higgins call for research in problem formulation. Besides being an important goal in its own right, problem formulation may help to make students better problem solvers, as the work by Keil (1964) suggests. In a study with some questionable procedures, Ammon (1972) investigated whether 170 fourth and fifth graders generated more problem statements after seeing a picture or after seeing a written description of the same theme (each was presented via a slide). Even if design difficulties are ignored, the statistically significant (0.01) results may not be of practical significance (pictures: 3.32 problems generated; written: 3.07 problems generated). Indeed, based on raw means the results favored the written versions in 8 of the 18 situations and the picture versions in 9 of the 18 (one situation gave equal means). Thus it is difficult to assert the superiority of either format. Even if substantial differences had appeared, it is likely that learners should have some practice in formulating problems in several formats--from words, from pictures, and from physical situations.

## Visual Imagery

As a result of several studies Paivio was willing to assert, "The most general assumption is that verbal and nonverbal information are represented and processed in distinct but interconnected symbolic systems" (1974, p. 8). Hence, different formats--e.g., pictures vs. words only--

might lead to different processing by different students. A drawing or physical object could supply information that a low-imagery student might not process from a word statement of a problem. Studies linking problem format, visual imagery, and brain hemispheric dominance (e.g., Wheatley, Mitchell, Frankland, and Kraft, 1978) would seem to be an area where problem-solving research can profit from basic psychological research. For example, Rohwer and Matz (1975) found that accompanying the reading of a passage by a picture resulted in better learning than accompanying the reading of a passage by a written version. Their 128 fourth graders gave performances favoring the picture treatment whether tested with the picture/printed version out of sight or within sight. Such work might suggest an improved format for presenting problems to some students.

## Some Recommendations

1. The relative effects of different formats for problems should be investigated, particularly as they relate to learner characteristics. Without considering learner characteristics a global look may be too gross, judging from the inconsistent results from studies of manipulative materials. The potential seems great. If, for example, a picture format is better for many students, it would be an easy matter for publishers to put more problems-via-pictures in textbooks.

2. What within-format variations make a difference? For example, if all the data are presented at the top of the page or in a chart or a picture, rather than in the tidy 3 or 4 sentence package, would students become more proficient at ignoring extraneous data? Bana and Nelson's (1978) observations that youngsters can be distracted by irrelevant aspects of concrete materials, as well as Campbell's studies mentioned earlier, show that within-format variations may be very important to children.

3. There is a nagging feeling that instruction which includes a variety of formats _should_ be more effective. Perhaps some payoff could be found in long-term retention of ability to solve similar problems or in attitude toward mathematics.

4. Krutetskii (1976) noted the mathematical "cast of mind" of students talented in mathematics--i.e., their tendency to impose mathematics on situations which most people regard as non-mathematical. Perhaps experiences in problem formulation with physical objects might result in a greater mathematizing on the part of average students.

## Language Factors in Routine Problem Solving

The role of language in the problem-solving process has received considerable attention as an area of research for many years, and a significant amount of general information has been accumulated. For example, the relatively high correlation between mathematical problem-solving ability and the ability to read and comprehend written material has been well established by numerous studies since the beginning of the twentieth century.

Several reviews of research in this area have shown correlations between reading ability and mathematical achievement to range from .40 to .86 (Monroe & Englehart, 1931; Aiken, 1972). Sizable correlations between problem-solving ability and reading ability have also been demonstrated. For example, Martin (1964) found that the correlation between reading comprehension and problem-solving ability, with computational ability partialed out, was higher for fourth and eighth grade students than the correlations between computational ability and problem-solving ability, with reading comprehension partialed out. Other studies have shown related results (Murray, 1949; Cottrell, 1967; Harvin & Gilchrist, 1970).

Following the suggestions of earlier investigators (Monroe & Englehart, 1931), efforts have been made to determine the more precise relationships between specific language factors and problem solving. To help classify these studies, it may be productive to consider the routine problem-solving process to be composed of several stages, each requiring a different level of analysis. It is in the first state that the problem solver must perform a surface analysis of the problem task, followed by a more detailed semantic analysis of the problem statement. It may therefore be convenient to classify language variables into two subtypes. Those that deal with the meaning of words and phrases, semantic variables, are actually special types of context task variables. Numerous studies have attempted to determine the relationship of vocabulary level, a semantic variable, to reading difficulty of routine problems and to instruction in problem solving. Variables that deal with the form of problem statements, their syntactic complexity, are important at the surface level of analysis. The term "syntax" will be used to denote those variables which account for the arrangement of and relationships among words and symbols in routine problem statements. It is clear from this definition that many syntax variables, particularly those involving sequencing of information and position of sentences and phrases are reflective of a problem's underlying structure and therefore directly affect the ease or difficulty of decoding and processing the information contained in problem statements.

While the distinction between semantics and syntax has traditionally been made in studies of language acquisition and reading, both types of variables are often present in studies of mathematical problem solving. Aiken (1972) reports that the data included in the 1963 Technical Report on the California Achievement Tests are representative of a number of findings which show that Reading Vocabulary (semantics), Reading Comprehension, Mechanics of English (syntax), and Spelling have sizable correlations with Arithmetic Fundamentals and even higher correlations with Arithmetic Reasoning.

A particularly interesting study designed to explore the relationship of difficult vocabulary and syntax to routine problem-solving ability was conducted by Linville (1970). Four arithmetic word problem tests were constructed. The problems in each were the same structurally, but varied according to difficulty of syntax and vocabulary. Fourth grade students (n = 408) were randomly assigned to one of four treatments: Easy Vocabulary, Easy Syntax; Difficult Vocabulary, Difficult Syntax. Significant main effects favoring the easy syntax and easy vocabulary tests were found. Not surprisingly, the investigators also found that in all four treatments, students of higher general ability and/or higher reading ability performed significantly better than students of lower ability.

While very few investigations have been concerned solely with the relationship of syntax to routine mathematical problems, the role of semantics has been studied extensively. A number of older studies (Hansen, 1944; Treacy, 1944) as well as more recent studies employing a linear regression model have shown that knowledge of vocabulary is an important factor in the ability to solve routine mathematical problems. For example, a study by Johnson (1949) revealed correlations of .45, .50, and .51 between tests of arithmetic reasoning and the Primary Mental Abilities Vocabulary Test. Arnold (1968) found the correlation coefficients for the relationships between knowledge of mathematical vocabulary and ability to solve word problems which did or did not contain mathematical terms to range from .38 to .78, from a sample of 167 sixth grade children.

With 296 sixth grade students, Early (1967) attempted to assess the effects of the presence or absence of the semantic variable word clues on performance on a 26 item test of routine verbal problems in mathematics. For the entire sample tested, students performed significantly better when word clues were present than when they were absent. Low performers were found to rely more heavily on word clues than middle or high level performers. This study is particularly significant since it is one of the few language investigations that considered correct choice of algorithms as a dependent variable. The above results held for both algorithms selected and correct solution.

## Readability

During the last decade, several attempts have been made to use the relationship of vocabulary and syntax variables to reading difficulty, as an index to classify mathematics materials. Several types of readability formulas have been used for English prose, and a few of them, particularly the Dale-Chall formula, the Spache formula, and the Cloze technique, have been applied to mathematics texts and routine problems. A number of investigations which have employed one or more of these formulas have demonstrated a wide range of readability levels in selected mathematics textbooks (Shaw, 1967) and have provided evidence that the readability level of mathematics problems can have a significant effect on problem-solving performance (Thompson, 1968). However, the application of readability formulas to mathematics materials and problems has not, as yet, been widely accepted as a defensible approach. Kane (1968, 1970) maintains that readability formulas for ordinary English prose are usually not appropriate for use with mathematical materials in several ways: (1) letter, word, and syntactical redundancies are different for English prose and mathematical material, (2) unlike ordinary English, the names of mathematical objects usually have a single denotation, (3) the role of adjectives becomes more important in mathematical English than in ordinary prose, and (4) the syntactic structure of mathematical English is less flexible than in ordinary English. Despite these claims, in a more recent study, Hater and Kane (1970) found the Cloze technique to be a highly reliable and valid predictor of comprehensibility of mathematical material designed for secondary students.

Very little information is available on the readability level of mathematics problems as compared to the average reading ability of students at each grade level. The few studies that have been done offer conflicting conclusions. After reviewing the literature on reading in mathematics, Earp (1969) concluded that the vocabulary of arithmetic texts is often at a higher readability level than the performance level of students in the classes where texts are used. He also noted that there is little overlap between the vocabulary of reading texts and that of arithmetic texts. Somewhat different results were reported by Smith (1971). After surveying the readability of sixth-grade arithmetic texts (as measured by the Dale-Chall formula), Smith found: (1) the average readability of routine problems fell within the normal bounds usually considered appropriate for that grade level, (2) the readability levels varied widely from problem to problem within the same text, and (3) overall the readability levels of the texts were generally comparable to those of related mathematics achievement tests. Smith concluded that readability may not be the most important factor in arithmetic problem-solving difficulty for this population of students. This conclusion, however, is based on the assumption that the Dale-Chall formula is an appropriate instrument to use with word problems in mathematics, an assumption that needs verification before these results can be meaningfully interpreted.

In the study mentioned earlier, Knifong and Holtan (1976) analyzed the written solution of 35 sixth-graders to word problems in the Metropolitan Achievement Test. They concluded that poor reading ability could not have been a factor in 52% of the problems, since errors on these problems were strictly computational or clerical. The role of reading difficulty for the remaining 48% of the mistakes was not determined.

## Instruction in Syntax and Semantics

Although the evidence is far from conclusive, it is still reasonable to assume that if problem solvers have difficulty reading a problem statement, they are less likely to be able to understand and solve it correctly than if they can read it with relative ease. If, indeed, syntax variables relate to, and therefore affect, the difficulty of verbal problems in mathematics, it seems reasonable to expect that instruction designed to help children deal with semantic and syntax variables could be effective in reducing the difficulty of many problems. Research on this hypothesis has been conducted for many years. Several early studies have shown that instruction on specific mathematical terms produces significant gains in problem-solving ability (Dresher, 1934; Johnson, 1944). Further evidence continues to be obtained in more recent studies. VanderLinde (1964) conducted an experiment with nine fifth grade classes matched with nine control classes on IQ and scores on achievement tests in vocabulary, reading comprehension, arithmetic concepts, and arithmetic problem solving. Classes in the experimental group studied different lists of eight quantitative terms each week for a period of 20-24 weeks. Results of the achievement tests following the 24 week period showed significantly greater gains by the experimental group on both arithmetic concepts and problem solving. Students with low IQ scores showed smaller gains than students with average or above average IQs.

At this point, it should be emphasized that any discussion of training in syntax must include a discussion of reading instruction, for it is clear that the ability to understand the meanings of words and the ability to process syntax is essential in learning to read all types of material (Aiken, 1972). However, as Henney (1971) notes, students often find reading mathematics to be different, and in general, more difficult than reading other materials. Spencer and Russell (1960) have pointed out that students experience difficulty in reading arithmetic material for several reasons: (1) the names of certain numerals are confusing; (2) number languages which are patterned differently from the decimal system are used; (3) the language of expressed fractions and ratios is complicated; (4) charts and other diagrams are frequently confusing; and (5) the reading of computational procedures requires specialized skills.

The question of whether reading instruction, particularly reading instruction in mathematics, can have a positive effect on the ability to understand mathematics and mathematical problems has only recently been investigated. Gilmary (1967) found that elementary school children who received instruction in both reading and arithmetic gained one third of a grade more on the Metropolitan Achievement Test - Arithmetic than did a control group which received instruction in arithmetic only. The results were even more pronounced favoring the experimental group when differences in IQ were controlled statistically.

In contrast, Henney (1968) tested the effects of 18 lessons on reading verbal problems with 179 fourth grade students. Approximately half of this group received the lessons over a nine week period. During the same time period on alternate days, the other half of the students studied and solved verbal problems in any way that they choose, under the direction of the same teacher. The results showed significant gains for both groups over the nine week period, but no significant differences were found between the groups on the verbal problem posttest.

A number of other studies relating to the effects of training in reading mathematical word problems have been reported, but many of these have suffered from lack of control or have been conducted with very few subjects. Nonetheless a few recent studies dealing with instructional techniques should be mentioned, since their results offer some promising suggestions for future research. Earp (1969) noted that verbal problems have a high conceptual density factor and include three types of symbolic meanings--verbal, numerical, and literal--within a single problem task. He maintains that three kinds of reading adjustments are required (that is, adjustments from the reading pattern used in ordinary English prose): (1) adjustment to a slower rate than with narrative materials; (2) varied eye movements, including some regressions; (3) reading with an attitude of aggressiveness and thoroughness.

A number of suggestions for helping students read word problems have emerged from the literature. Earp (1970), for example, has suggested five steps in reading verbal problems:

(1) Read first to visualize the overall situation.
(2) Read to get the specific facts.
(3) Note difficult vocabulary and concepts.
(4) Reread to help plan the solution.
(5) Reread the problem to check the procedure and solution.

The effectiveness of the use of the above five steps was tested by Barnett (1974). In this study, Barnett defined a composite variable composed of key mathematical terms and two measures of length (defined in terms of the number of words and sentences in the routine problem statements). In an attempt to show that such a composite semantic and syntactic variable could be used as a basis for writing instructional material, Barnett wrote an instructional unit designed to help students overcome the difficulty attributable to this variable, using some of Earp's suggestions with certain modifications. The subjects consisted of 150 pre-service elementary education majors randomly assigned to either an experimental group (which received the instructional unit) or to one of two control groups (which received an instructional unit not related to problem solving). The results showed that the experimental group made significantly greater gains than either of the control groups. Posttest scores for the two control groups were not significantly different from each other. The results suggest that variables of this nature may be effective as a basis for writing instructional material to help students learn to read routine verbal problems with understanding.

Several other attempts have been made to design instructional procedures to help children read mathematics materials and problems. Taschow (1969) suggests a remedial-preventive program in reading mathematics. Students are first given a Group Informal Reading Inventory to determine which students have difficulty with reading mathematics. In the second phase, a five step program called the Directed Reading Activity in Algebra is given to each child. The five step DRA consists of: (1) readiness, (2) guided silent reading, (3) questions, (4) oral reading when needed, and (5) application. While this program does not provide instruction in specific semantic or syntactic variables, the exposure to, and practice with, reading mathematical materials can help students learn to deal with the more difficult syntax structure and vocabulary found in routine verbal problems.

Another program offers more specific instruction in processing syntax structure. Dahmus (1970) suggests a "direct-pure-piecemeal-complete" (DPPC) approach to solving verbal problems. In this method, the student is encouraged to translate the data presented in the problem into mathematical sentences, by concentrating on a few words at a time. He gradually learns to put together the "piecemeal" mathematical statements into equations, and finally, into systems of equations. It is clear that the ability to translate data in routine problem statements into mathematical form is one of the most important aspects of general problem-solving ability. It seems that it is also one of the most difficult abilities to cultivate. Several procedures, similar to the one above, have been suggested, but it is not at all clear if any of these procedures are effective across populations and problem types.

Although the results of some of the previously discussed investigations are encouraging, it is evident that to understand the importance of reading difficulty to problem-solving ability, researchers must address themselves to determining what specific components of verbal problem readability (such as grammatical structure, vocabulary, etc.) affect problem-solving behavior, and how the roles of these factors change over different age groups and problem sets. A number of recent studies employing a linear regression model have demonstrated a potentially fruitful method for investigating these questions.

## Regression Studies

The first studies applying the multiple linear regression analysis to mathematical problems were conducted at Stanford University with elementary school children, operating in a computer-assisted instructional mode (Suppes, Hyman, & Jerman, Note 2; Suppes, Jerman, & Brian, 1968). The problems studied were computational arithmetic problems, involving one or more of the four basic arithmetic operations. Since these "pioneer" studies did not directly deal with routine verbal problems, the reader is referred to other sources for a discussion of the procedures used in these investigations (Loftus, Note 3; Segalla, Note 4; Barnett, 1974).

Encouraged by the results of the first few studies, a first attempt to extend the linear regression model to mathematics word problems was made by Suppes, Loftus, and Jerman (1969). In this study, 68 word problems were presented and solved in a computer-assisted instructional mode, using 27 above-average fifth grade students. The syntax variable of LENGTH (a measure of the number of words in the problem statement) was used for the first time, along with several other variables dealing with the number of operations used to reach a solution, the sequence of data, and the semantic variable of verbal cues. The results were disappointing. The semantic and syntactic variables did not significantly affect problem difficulty, and the three variables that were found to be significant accounted for only approximately 45 percent of the variance in problem difficulty. However, the organization and procedures used in the study provided a model for further investigations to use and refine.

A 1970 study by Loftus (Note 3) used the six variables from the previous study, plus two new ones, the syntax variables of ORDER (indicating the sequence of data presented in the problem) and DEPTH (a measure of grammatical complexity). A set of 100 problems was administered to 16 sixth grade students, characterized as "low ability". The students solved the 100 problems after four weeks of practice on a computer teletype. The results showed an $R^2$ value (the amount of variance in proportion of problems done correctly) of .70, a respectable amount of variance accounted for. Both the syntax variables of DEPTH and LENGTH made significant contributions to the amount of variance accounted for, and entered the regression equation in the third and fourth steps respectively. It should be noted, however, that the number of subjects was very small and it was assumed that using the partial correlation coefficients for the variables was a valid measure of importance.

In the following year, a number of studies provided further evidence of the consistency of some of the previously defined variables, defined new ones, and extended the mode of presentation to paper-and-pencil. Jerman (Note 5) reported the results of two studies. In the first, Searle re-analyzed the data from the 1969 Suppes, Loftus, Jerman study, using 14 new variables, including an ORDER variable. Both the ORDER and LENGTH variables were found to be significant. Jerman followed up this study with an investigation using 30 word problems administered to 20 fifth graders in a paper-and-pencil mode. Five variables, including the syntax variable LENGTH, were found to account for 87 percent of the variance in problem difficulty. Further support for the LENGTH variable was found in a study by Jerman and Rees (1972) and in a follow-up study by Jerman (Note 6).

At this point it should be noted that direct comparisons of the importance of variables from one study to another are no longer possible. Investigators began to modify the definitions of the variables in each study, and used different problem sets and various grade levels. More recent studies, however, have attempted to show similarities between the variables and by observing the trends, have tried to generalize results to several problem types and grades.

After six years of experimentation with variable definitions and the linear regression equations, the time seemed right to apply these previous results predictively. Using the data from Jerman's 1971 study with students in grades 4 to 9, Jerman and Mirman (1974) took the top six variables found in that study and coded them on a new problem set. Using the resulting regression model, they then attempted to predict before administering the problem set, the proportion of students in a new population that would correctly solve each of the problems. The results indicated that the regression model based on data from grades 4 to 9 was unsatisfactory. The data were then re-analyzed, with the same six variables used for each grade level. The resulting regression equations for each grade level gave much better predictions, with residuals of percent correct ranging from 4 to 15 percent. Although these results were not as good as the researchers would have liked, the study did establish a model for further investigations. It remains for future investigations to contribute to the understanding of language variables, so that predictive equations can be refined to yield results in an acceptable range.

The application of the regression model to arithmetic word problems was extended to the junior college population by Segalla (Note 4). Convinced of the importance of syntax variables, Segalla defined thirty variables that included many syntax variables not defined in any previous study. A set of 172 word problems was administered to 44 low ability junior college mathematics students. Based on the size in the drop in $R^2$ when the variable was removed from the regression equation, the set of the six most significant variables included the syntax variables of ORDER, NOUNS (the number of nouns in the problem statement), DEPTH, LENGTH, and ADVERBS (the number of adverbs in the problem statement).

As interest in the regression model began to grow, it became apparent that syntax and semantic variables played an important role in determing problem difficulty for subjects of all ages. In 1973, Krushinski investigated the relative importance of 14 syntax variables with respect to each other. The set of variables included eight dealing with aspects of length, four dealing with grammatical structure, and two dealing with numerals and the position of the question sentence. Three sections of pre-service elementary school teachers enrolled in a course in the teaching of arithmetic were administered a problem-solving test. The amount of time permitted on the test varied from 20 minutes, to 60 minutes, to one day for the three sections. Krushinski found that six variables, NUMBER OF SENTENCES, NUMBER OF CLAUSES, CLAUSE LENGTH (the average length of the main clauses), NUMBER OF PREPOSITIONAL PHRASES, NUMBER OF WORDS IN THE QUESTION SENTENCE, and NUMERALS IN THE QUESTION SENTENCE entered the regression analysis within the first six steps in at least two of the three sections. After the sixth step, the multiple R's for the three sections in order of decreasing time limits were .856, .738, and .626. In light of these interesting results, it is tempting to speculate that as time becomes a crucial factor, some syntax variables decrease in importance with respect to other factors. However, there is not enough evidence to generalize these results with any degree of confidence.

Following the Krushinski study, Beardslee and Jerman (1973) attempted to apply Krushinski's 14 syntax variables to a problem set appropriate for students in grades 4 to 8. Three test forms of 30 problems each were prepared using a problem set from a previous study. The number of words was systematically varied, so that two forms contained one-third more and one-third fewer words than the original problem set. Eighteen separate analyses were conducted on the data. Only two of the six variables which Krushinski found to be significant entered consistently among the first six variables in the linear regression analysis. In addition to these two variables, CLAUSE LENGTH and PREPOSITIONAL PHRASES, two other variables, SENTENCE LENGTH and WORDS IN SUBCLAUSES entered the regression consistently within the first six steps on two or more test forms. These results suggest that it may be possible to identify syntax variables that are important for both college and pre-college students. This study is significant in that it is one of the few attempts to observe the effects on problem-solving ability resulting from systematic variations in syntax.

Using the same data set from the previous study, Beardslee and Jerman (Note 7) conducted a second study with the same population of students from grades 4 to 8. Once again, the number of words was systematically varied on three test forms. One purpose of this second study was to extend the previous study to include syntax variables not used in the 1973 Krushinski study. A second purpose was to investigate a wide variety of measures of problem length, to determine which definition accounted for the most variance in proportion of problems correct. This second purpose was of particular importance, since the many definitions of length employed in several different studies made interpretation of results extremely difficult. Seventeen variables were defined for the investigation. Of these, nine were considered to be different versions of the problem length variable. Although none of the variables was found to account for a significant amount of variance for all grades, five of them were significant for several grades on one or more test forms. Most interestingly, not one of the nine variations of the length variable was shown to be superior to any of the other length variables. It would appear that the many definitions of length really describe the same thing.

Although most of the linear regression studies included a number of types of variables, the dominance of any particular type of variable (structural, computational, syntactic, etc.) was not established in terms of the importance of the variable type in determining problem difficulty. This question was investigated by Beardslee and Jerman (Note 8) in a study involving five structural variables, and twelve topic variables. A 50-item achievement test was administered to fourth and fifth grade students. After a regression analysis involving only the twelve topic variables, four were selected to be combined with the five structural and four syntax variables. The results showed that three variables made significant contributions to the amount of variance accounted for; the topic variable GEOMETRY, and the two structural variables dealing with multiplication and cognitive level. None of the syntax variables was found to be significant, and the total amount of variance accounted for was only .47. Despite these disappointing results, the study established the need for a more inclusive model. As the experimenters state, "None-the-less some encouraging signs seem evident. One, that a combination of different classes of variables produces a higher R than using only one class" (Beardslee & Jerman, Note 8, p. 10).

## Some Recommendations

1. Researchers in the field of mathematics education should continue to try to make use of new developments in the field of linguistics (such as the syntax complexity formula developed by Botel, Dawkins, and Granowski,

1971), and work with investigators in this area to apply their methods, formulas, and definitions to mathematical problem solving.

2. While a large variety of semantic and syntax variables have been identified, very few studies have been conducted to determine their contribution to a better understanding of routine problem-solving behavior. The relationship of specific syntax and semantic variables to the other aspects of verbal problem tasks, such as format, problem structure, and heuristics, need intensive study across age and ability groups. Of particular importance are studies that attempt to determine the role of syntax and semantic variables in the decoding process in the first stage of problem solving.

3. Another area of needed research is that concerned with the improvement of instruction in reading and its relationship to improved problem-solving ability. The application of existing methods of determining readability to mathematics word problems is a good beginning, but perhaps what is really needed are formulas or techniques that are based on semantic and syntactic parameters specific to mathematical word problems. Although extremely little research has been conducted on the effects of variation of syntax and semantics on readability and comprehensibility of mathematical problems, this is precisely the type of research that is needed to form a basis for designing instructional material. The 1970 study by Linville discussed previously is a good example of this type of research.

4. While the results to date of studies that have used a linear regression model are far from conclusive, some consistencies have already emerged in terms of which general categories of variables need to be controlled or varied systematically. However, from the large variety of subjects, problem sets, and definitions, it is evident that improvements in this area are needed. First, knowledge of the relationships of semantic and syntactic variables to problem difficulty will be increased only when studies can be compared meaningfully. Investigations using comparable populations would be helpful. More comprehensive studies across many grade levels would also help establish which variables are the most important for each level of development. The use of similar problem sets would provide control of a number of important variables. A major problem in this area of research has been the many different definitions of the same variable, making comparison of results across studies impossible. More studies like the 1973 Beardslee and Jerman efforts are needed to discover which definition is best for each semantic and syntactic variable. Once this information is obtained, studies designed to investigate the relative importance of various categories of variables need to be conducted.

5. While the linear regression model has shown some promise as a research technique in the area of structural and language variable research, it is clear that in its present form it falls far short of being able to predict problem-solving success. Improvements in the model might include different criteria of importance, such as the size of the regression coefficient, size of the partial correlation coefficient, order

of entry into the regression equation, contribution to the total variance, and size of the drop in $R^2$ caused by removing the variable from the analysis. Until more information is obtained to give direction to the choice of the best factors to use in the model, it would be helpful for studies to provide data on several dependent measures used with several measures of importance. Of course, studies should be replicated to insure stability of results for meaningful interpretation.

6. It is interesting to note that in almost all of the studies concerned with verbal problem solving, the dependent variable is the "correctness" of the solution or the latency of response time in reaching the correct solution. This author could find only one study (Early, 1967) that considered variation of semantics as the independent variable and "process" used (selection of algorithms) to find a solution as the dependent variable. No studies could be found that considered changes in processes used as a result of changes in syntax. The effects of language factors on problem-solving behavior--i.e., "process" in the modern sense--is clearly an area of research that has been overlooked for too long. It is unreasonable to believe that a model of verbal problem-solving behavior will ever be developed, unless process as well as product is taken into consideration. The role of process variables will be considered in the next section.

## Process Variables

Research on mathematical problem solving has deservedly been labeled one of the most chaotic of all areas of research. It is therefore understandable that within the research on problem solving, a specific focus such as process variables would be fragmented and often contradictory. Beginning in the early 1920's until the present day, problem-solving research admits to the importance of process variables but usually bases the conclusions on the correctness of the solution or the product of the problem-solving activities. Researchers today agree that the process of problem solving is a collection of moves or activities which focus the search for the correct solution and the product of problem solving is the actual solution to the problem.

This review of problem-solving process research is separated into two distinct periods of time. Most of the studies reported before the middle of the 1940's considered "process" variables as meaning the fundamental operations of addition, subtraction, multiplication, and division. These studies will form the first part of this section, even though this view of "process" is more akin to "product." From the middle 1940's until the late 1960's investigators tended to report results based only on product scores or on the number of students who solved the problem versus the number of students who did not solve the problem. Therefore this review will not reflect the period of time from 1945 to 1965. (As a case in point, Spitzer and Florney (1956) examined five different textbooks for that time period and could not fine one single specific procedure for improving problem solving.) Since 1965, however, many studies have been reported that address the problem-solving process directly. These studies will form the second part of this section.

## Earlier Studies

Success in solving multi-step routine problems can be described in terms of the basic operations involved. Berglund-Gray and Young (1940) studied the influence on the difficulty of solving two-step problems when the order of the basic operations is reversed. For example, the study considered the difficulty of these two problems.

> Mr. Smith bought 24 melons for $2.88 and sold them at a profit of $3.12. For how much did he sell each melon?
>
> Mr. Smith bought 24 melons for $2.88 and sold them at a profit of $.13 apiece. For how much did he sell each melon?

Two tests were constructed, using the same problems written with the order of the basic operations reversed, as in the examples. The criteria were the interpretation or understanding of the problem and the recognition of the basic operations needed for the solution. It was not necessary for the 5th, 6th, and 7th grade students to solve the problems. Since all the problems were two-step problems, an answer sheet was devised on which the students indicated the order of the basic operations they would use if they actually had to solve the problem. Examples from the tests show the type of problems as well as format of the answer sheet.

> Alice picked 13 quarts of cherries. Her sister picked 6 times as many. How many quarts did both pick together?

|   | A | S | M | D |
|---|---|---|---|---|
| 1 |   |   | x |   |
| 2 | x |   |   |   |

> John's mother gave him 45¢ to buy bread. He lost 10¢. How many loaves of bread can he get for the money he has left, if bread costs 7¢ a loaf? (Remember these were 1940 prices!)

|   | A | S | M | D |
|---|---|---|---|---|
| 1 |   | x |   |   |
| 2 |   |   |   | x |

Conclusions from the study are quite clear: the order of occurrence of the basic operations in two-step problems does affect the difficulty of solving these problems. It was found that division followed by any of the other three operations gave the most difficult problems. Also, the order of subtraction-addition and subtraction-multiplication was more difficult than their reverses. Interestingly, the order of addition-multiplication was shown to be more difficult than multiplication-addition. Throughout this study the term "process" was used exclusively to identify the basic operations in arithmetic. Processes and basic operations needed to solve the problems were identical.

Since the basic operations were considered the most important "process" variable before the middle 1940's, John (1930) focused exclusively on computational or algorithm errors made in solving two-step arithmetic word problems. This study with intermediate grade students identified forty distinct errors within the categories of reasoning, fundamentals, reading and miscellaneous. A problem such as the following was solved and a classification scheme of common errors was devised.

> The lunch counter sells apples. In the last box there were 140 apples. Six of them spoiled, and the rest were sold at 5¢ each. How much money was received for the apples?

Some of the errors in reasoning were the use of a wrong process--e.g., addition for subtraction--selection of a process by numbers or by the types of numbers on a problem, and combining unlike quantities. Some of the errors in fundamentals were the failure to attempt multiplication or division, the inability to interpret a fraction, and the attempt to subtract three numbers. Recently Hutcherson (1976) conducted a partial replication of the John investigation. This study determined the types of errors that sixth grade students make in solving routine two-step word problems. Hutcherson concluded that there had been no major change in the list of errors students make today when compared with forty years ago. However there was some evidence that students today make errors more frequently than their counterparts did in the 1920's. According to the investigator some of this difference could be attributed to the lack of exposure of today's students to these types of problems.

During the 1920's two major studies addressed the issue of teaching students how to increase their problem-solving ability by emphasizing the process involved in solving a word problem. In a series of studies, Washburne and Osborne (1926) found that there was no definite relationship between the ability to perform the formal analysis of problems and problem-solving ability. (Formal analysis meant the identification of the "given " and the "to find" terms, the elimination of extraneous data, the selection of the basic operation needed, and estimation of the result.) Their general conclusion was interesting: they suggested that merely giving many problems for practice without any instruction in analysis was the most effective method. In other words, practice in, or exposure to, problems was more effective than trying to teach a definite process procedure. Newcombe (1922) had found the opposite to be true. His study forced an elaborate scheme of steps that had to be completed before the problem was solved. A group of students was taught very specific guidelines while another group had regular classroom instruction. The group given instruction in specific guidelines had to work every problem in the following format.

> If a motor truck delivered 91 tons of coal in one day at a cost

of $18.24 and a team and wagon delivered 40 tons in one day at a total cost of $15.84, find the amount saved on each ton by delivering with a motor truck.

Given: 91 tons
$18.24 cost by truck
40 tons
$15.84 cost by wagon
To Find: amount saved per ton
Processes: division, division, subtraction
Approximate answer: 20 cents
Solution:
Check:

Results showed the group given instruction in specific guidelines was superior in speed and accuracy when compared with the group receiving only regular classroom instruction. This study supported the idea that it is possible to teach students part of the process of problem solving.

## More Recent Studies

The second section of related research in process variables will review studies since the middle of the 1960's. Without a doubt, the greatest influence on these studies has been the writings of George Polya. Heuristics, heuristic strategies, and process sequence variables are terms mentioned in recent studies concerning problem solving. Polya's influence unfortunately has not lessened the unpatterned and fragmented nature of problem-solving process research. In some respects it has heightened the confusion since the terms have not been defined or distinguished by the researchers. Information processing proponents claimed they would sort out the confusion. Again this claim has not been fulfilled at present. However, Damarin (1976) applied concepts from the psychological study of problem solving and determined that three elements from Polya's list--understanding the problem, carrying out the plan, and looking back--are important in the process of solving problems. Psychological research does not support Polya's fourth element--devising a plan--but rather supports the idea that this aspect is really restructuring of the data. Silver (1977) has explored one aspect of the devising-a-plan element: think of a similar problem. Many of his eighth-graders classified problems as being similar on the basis of some shared measurable quantity (e.g., time or age). After seeing solutions, however, more students used associations based on underlying mathematical structures.

A few studies during the 1960's were not directly influenced by the Polya emphasis but rather were reflective of the studies conducted during the 1920's. As noted earlier, a study by Early (1967) attempted to determine whether the absence or presence of word clues had any effect on the correct process to solve the problem. These sixth grade students selected the correct algorithm to solve word problems more frequently if the problem had word clues. Also more practice with word problems seemed to cause the dependence on word clues to lessen considerable. Another study by Lerch and Hamilton (1966) detected two distinct categories in the ability to solve problems, the ability to program or to determine the correct procedure and the ability to process or to do the correct computations. By receiving specific instructions in writing a number sentence which totally described the problem, students were able to improve their ability to determine the procedure to be followed in solving problems. The ability to program was found to be more important than the ability to perform computations.

Research studies during this time period gradually moved from a fixation on arithmetic processes to more general processes. When this transition took place many studies classified process variables into broad categories or strategies. An investigation by Sanders (1973) concentrated on discovering what strategies successful and unsuccessful fourth graders used in solving routine textbook problems. The most popular strategy was Logical Analysis which meant solving the problem with an equation or algorithm. The most successful strategy was Creative or Divergent Thinking which meant solving the problem in unusual ways or suggesting many possible solutions. The least successful strategies were Blind Guessing and Trial-and-Error. Successful problem solvers usually employed Logical Analysis or Creative Thinking as their strategies while unsuccessful problem solvers usually employed Blind Guessing as their favorite strategy. A closely related study by Hollander (1974) with sixth grade students also identified successful problem-solving strategies. Conclusions showed that successful problem-solving strategies reflect the ability to reason analytically, reason insightfully, and compute the solution in the minimum number of steps necessary. Surprisingly, successful problem-solving ability was not related to accuracy in recalling information in the problem or even remembering what information was asked for within the problem.

A study by Means and Loree (1968) identified three process areas: retrieval or recall of information necessary to solve a problem, extraction of information in the problem, and combining operations or strategies. The results indicated that instruction had influence only in improving the abilities to extract information and to combine strategies. Instruction did not improve the ability to retrieve information. It was believed that the effects in increased problem-solving ability would be noticed only after a prolonged period of instruction in these three process areas.

Components of a model of the problem-solving process were listed by Robinson (1973). These components were the problem, previously learned content, previously learned processes, search, organization, and a solution. Within the model the following aspects were considered important: recognition, analysis, search for related processes or content, organization of processes or content, and synthesis of a procedure for arriving at a solution. Consistency of strategy was not evident for individual students from problem to problem nor were the strategies similar from problem to problem. Successful problem solvers devised a formal strategy and took more time to solve the problems. Unsuccessful problem solvers approached the problem with many different strategies and tended to frequently use the strategy of trial-and-error. Another study by Webb (1975) does not support the contention that a range of strategies is a characteristic of a poor problem solver. This study identified the better problem solver as one who used a range of strategies and a medium amount of trial-and-error.

Shields (1976) adopted an information processing approach to problem solving. Strategies exhibited by fourth grade students were examined to see if they coincided with an information processing model. This model has four distinct steps: 1) clearly define the goal, 2) generate alternatives, 3) synthesize information concerning these alternatives, and 4) choose the result which is most consistent with the goal. Problem-solving processes of the model were evident when the students attained accurate solutions.

A particular strategy, trial-and-error, seems to appear in many different studies and yields inconsistent conclusions. Some studies claim trial-and-error as the most effective strategy while other studies imply that it may be harmful to encourage trial-and-error as a strategy. Maxwell (1975) found that trial-and-error was used frequently in initial approaches to a problem task but decreased in importance as the solution was approached. Also the continued use of the trial-and-error strategy lengthened the solution time and was identified as one of the main characteristics of ineffective problem solvers. Poor problem solvers tried a trial-and-error strategy more often than good problem solvers in the study by Robinson (1973). Grady (1976) established there was no difference between successful and unsuccessful solvers in the use of the trial-and-error strategy with routine algebra problems. At least two modes of thinking were evident in an investigation by Dalton (1975). These modes were deduction and trial-and-error, with the more effective problem solvers using the trial-and-error strategy. In a study by Kilpatrick (1968) deduction was applied more frequently than trial-and-error. Deduction as a strategy usually led to an incorrect solution while trial-and-error usually resulted in a correct solution. The group of students who applied the trial-and-error strategy least often also had the greatest difficulty with the problems, spent the least time in solving the problems, and had the fewest number of problems correct.

Since the trial-and-error strategy implies the ability to estimate, two studies have focused on the relationship of estimation and trial-and-error. Students with good estimation ability were superior in problem solving in an investigation conducted by Hall (1977). The relationship of estimation ability and problem-solving ability by trial-and-error was studied by Paull (1972). Results showed that the ability to estimate numerical computations was a significant predictor of the ability to solve problems by the trial-and-error strategy. Also the efficient use of trial-and-error in solving problems was not related to making accurate first estimates.

## Instruction in Processes

It is quite natural that following the identification of problem-solving processes, there would be research studies on the effect of instruction with these processes. With few exceptions, in the bulk of this type of research, instruction has not been effective in promoting increased problem-solving ability. A general model of instruction in problem-solving procedures was constructed by Post and Brennan (1976). This model had the following components: general heuristic problem-solving procedure; recognition, clarification and understanding of the problem; plan of attack-analysis; productive phase, validating phase-checking-proving. The conclusions of this study suggested that instruction in a model with general components of the problem-solving process was not effective in promoting increased problem-solving ability. In an earlier study Post (1968) also arrived at this same conclusion with younger students. In an exploratory study involving nonroutine problems, however, Lee's fourth-graders did use heuristics more frequently and solved more problems after 20 instructional periods than a control group (1977).

Rather than focus on general instruction in problem-solving procedures, Vos (1976) concentrated on five specific processes. The specific processes for instruction were drawing a diagram, approximating and verifying, constructing an algebraic equation, classifying data and constructing a chart. Results indicated that it was possible to teach the use of the processes but the increase in the ability to solve problems was very slight. Nelson (1975), cited earlier, investigated the effectiveness of teaching only one process-drawing a diagram. Conclusions from the study noted that students were better able to solve problems presented in diagram form but the treatment did not significantly affect their problem-solving ability. Evidence showed they did use the diagram method quite frequently in attempting to solve the problems.

## The Measurement of Processes

Research in the instructional effects of problem-solving processes is somewhat hampered by the crude coding schemes and instruments available. In addition, to analyze thought processes of a problem solver, interviews or ingenious written testing instruments are presently required. These instruments either involve complex coding schemes or oversimplified categories for which the data analysis is somewhat meaningless.

A technique developed by Rimoldi (1955) has been utilized by very few studies in mathematical problem solving. This technique involves giving a problem with a set of cards that have one question on each card and an answer on the reverse side. The investigator records the sequence of the questions selected to solve the problem. This sequence will indicate the successive steps in the solution of the problem. It is possible to define three different item properties for each card question. The utility index (1.00 to .00) for any one card is the ratio of the number of times that it has been selected and the total number of students. The median value of each card's appearance in time indicates during which part of the solution process the card was selected. Another item property is the dispersion of the cards, that is, certain questions are always toward the conclusion of the solution process. There are three different ways to score this instrument. A criterion group selects the cards and this selection is defined as the optimal sequence. An agreement score can be obtained by using the tau coefficient which estimates the agreement between the student's sequence and the optimal sequence. The higher this value the better the agreement. Another scoring scheme involves the utility index. A student's utility score (1.00 to .00) is the sum of the utility indexes of the selected cards divided by the total number of cards selected. A score can also be based on the number of cards selected. A comparison can be made with the average number of cards used by the criterion group with the number of cards selected by the students.

Several studies have devised coding schemes. As an example, Flaherty's 1975 system contained these seventeen variables:

misreads problem,
rewords problem,
draws diagram,
indicates familiarity with type of problem,
notes need for auxiliary information,
lacks a systematic approach,
recalls definition or auxiliary information,
fails to use correct auxiliary cues,
unsuccessful, adopts new approach,
fails to retain model of solution,
makes computational errors,
indicates concern about method,
signifies inability to solve problem,
uses equations,
uses deductions and arithmetic,
stops without solution,
makes structural errors.

Forcing the student to think aloud did not significantly influence the problem-solving score. However, the students who were required to think aloud made significantly more computational errors than the students not forced to verbalize their thoughts.

Another popular scoring-coding scheme was utilized by Kantowski (1977) in an exploratory study of processes in problem solving. This scoring-coding scheme gave one point for each of the following: suggesting a plan of solution, persistence, looking back, absence of structural errors, absence of executive errors, absence of superfluous deductions, and correctness of result. Presently there are some different scoring-coding schemes being developed but none of them can be considered either elegant or efficient for use with a large collection of data.

Barnett, Vos, Sowder

## Some Recommendations

Process variable research may be at the brink of a breakthrough if the following can be resolved:

1. Agreement among researchers on the definition of crucial terms such as process, heuristics, organizers, strategies, and behaviors.
2. Development of a scoring-coding scheme that is both elegant and efficient.
3. Development of test instruments that emphasize process variables rather than just reflect process variables.
4. Acceptance of the research profession for small-scale experiments that rely on protocol analysis techniques.
5. Increased cooperation among researchers interested in problem solving so that common problems, similar instruments and shared data analysis can be more easily facilitated.
6. More clearly defined techniques for instruction in improving problem-solving ability.

Optimism by researchers in problem solving is great at the present time. This optimism is justified considering the quality of research conducted and reported in problem solving since 1968. Problem-solving process research has only begun coming into its own since 1973, but already techniques and process variables are being used effectively in additional research studies as well as in classroom instruction.

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# Mathematical Learning Disabilities: Considerations for Identification, Diagnosis, Remediation*

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Why is a paper on "learning disabilities" being included in a monograph on "applied problem solving"? Traditionally, a great deal of the research on problem solving has focused on identifying and analyzing the abilities and processes used by bright students and good problem solvers. The assumption has been that average and below average ability youngsters can be taught to use the processes used by gifted problem solvers. Krutetskii (1976), however, has shown that this assumption may be unwarranted. That is, many of the abilities and processes used by gifted youngsters may be inaccessible to average ability youngsters. These inaccessible abilities may involve, among other things: (a) a "mathematical cast of mind"--a tendency to interpret the world mathematically, perceiving and remembering the mathematical structure of problem situations, (b) the ability for rapid and broad generalizations--often from single situations, (c) the ability to curtail reasoning processes radically--skipping intermediate steps in normal chains of reasoning, moving rapidly from problems to solutions.

In many ways, Krutetskii's work can be compared/contrasted with that of Piaget. For example, Piaget showed that young children are not just shrunken adults. The knowledge structures and thought processes of young children are qualitatively different (not just quantitatively fewer) than those of adults. Similarly, Krutetskii's research suggests that normal problem solvers are not just inferior versions of gifted problem solvers. The thought processes and knowledge structures used by gifted students may not simply be refined or elaborate versions of those used by normal problem solvers; they may be qualitatively different. Furthermore, the qualitatively different systems of thought used by gifted problem solvers may be just as inaccessible to normal problem solvers as formal operational reasoning is to most seven-year-olds.

The above argument does not imply that research about gifted problem solvers is irrelevant to the training of normal students. However, it does suggest that getting normal children to perform like gifted problem solvers (if this is possible or desirable) may not occur through the teaching of isolated skills, abilities, heuristics, or problem-solving strategies. Rather, students may need to develop a whole new mode of thinking--a transformation analogous to the qualitative shift from concrete operational to formal operational modes of thought.

If Krutetskii's conclusion is correct concerning the relative inaccessibility of many of the abilities used by gifted students, Piaget's research suggests another fruitful direction for research to take.

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Researchers can analyze the disabilities of poor problem solvers or the difficulty-causing processes for "learning disabilities" students and can investigate the extent to which these processes and disabilities also cause difficulties among average ability problem solvers.

Research on exceptional students, whether it involves "gifted" or "learning disabled" youngsters, serves important functions for theory development. The abilities of normal youngsters can be put into perspective by comparing their systems of thought processes to: (a) a less powerful state of thought processes used by poor problem solvers or (b) a more powerful system of thought processes used by gifted students. In some cases these less powerful, or more powerful, systems of thought may correspond to states of cognition from which normal problem solvers may have evolved or to which they may develop. But, even if developmental inferences are inappropriate, learning theorists can benefit greatly by comparing the cognitive abilities of normal students to those of exceptional students. Many assumptions about learning and problem solving seem quite sensible when they are only used to describe average ability youngsters, but may become obviously absurd when then are applied to exceptional children.

Unfortunately, investigating the mathematical abilities of learning disabilities youngsters seems to pose practical problems that are more severe than those which arise when investigating the abilities of gifted youngsters. Certainly mathematical abilities research involving learning disabilities students has been far less popular than that involving gifted students. Part of the difficulty results from the fact that investigating the development of mathematical knowledge and abilities in children requires two diverse kinds of expertise: (1) knowledge about the characteristics and capabilities of children and (2) knowledge about the mathematical concepts under investigation. But, few researchers have developed sufficient expertise in both of these specialized areas. Consequently, researchers who are knowledgeable about child psychology (particularly in special areas like "learning disabilities") have tended to avoid issues involving mathematical content; and researchers with expertise in mathematics have tended to focus on children who are most like themselves--mathematically gifted.

This paper consists of four parts. Problems involved in the identification, diagnosis, and remediation of LD subjects are discussed in Part I. Some important abilities and processes that LD subjects may lack are described in Parts II, III, and IV, beginning with three different psychological theories that are familiar to mathematics educators. The goal is to clarify some important problem-solving processes that are accessible, but perhaps not well developed, in average ability youngsters. The discussion should also allow some inferences to be drawn about methods of diagnosing certain types of disabilities and about some appropriate types of remedial activities. Hopefully, it will also encourage LD specialists to investigate problems involving mathematical abilities or encourage mathematics education specialists to investigate issues involving LD subjects.

## I. Learning Disabilities in Mathematics

It is common to meet unusually intelligent and highly educated people who claim to lack a "mathematical mind." The presumption is that regardless of a person's stock of prerequisite knowledge and prior training, there are certain underlying abilities that are unique and specific to mathematics, and that people who lack these abilities (even though they may perform outstandingly in other areas of thought) may be restricted in their ability to reason mathematically. But what are these mathematical abilities? How are they related to other categories of abilities (e.g., verbal abilities)? How can mathematical abilities be improved or refined? What compensating strategies can be mastered that might be helpful to people with mathematical "learning disabilities?" These are among the major issues to be considered in this paper.

Among "learning disabilities" specialists, there has been a great deal of controversy about the definition of LD subjects. From the point of view of a mathematics educator, there is even more ambiguity about the nature of the mathematical abilities that LD subjects presumably lack. Consequently, it is often unclear whether teachers should try to teach LD youngsters using techniques that are qualitatively different from those they use with normal children (or normal slow learners), or whether they should simply use the same techniques--only do it better, perhaps with more practice and drill.

### Learning Disabilities Subjects

Among specialists in the area of learning disabilities, there is general agreement that the field includes people of average or above average intelligence who exhibit failure in certain isolated areas, despite overall competence in other areas. Consider the following cases:

> Johnny (fifth grade): Johnny is one of the best students in his class in verbal and reading skills, but he is unable to carry out even the simplest arithmetic computations. He has difficulty judging quantities in everyday situations--e.g., how much food can go on a spoon, how much milk can go in a glass, how fast the milk should be poured, etc. He becomes easily disoriented in supermarket-type situations, and he has great difficulties with simple mathematical relationships--e.g., he has difficulty lining up buttons on his clothes. Consequently, he has difficulty in everyday situations where he must dress himself or eat without making a mess.

> Bill (seventh grade): Bill is very good at computation. He can add in his head a sequence such as 781 + 432 + 394 and always get the correct answer. His reading skills are also among the best in his class. He does, however, have difficulty with word problems. He can relate the relevant information given in the problem, but cannot

choose the correct operation to solve the problem. His difficulty is characterized by adding when he should multiply, dividing when he should subtract, etc. In other words, he cannot apply his good computational skills in a problem-solving situation. His disability is in mathematics reasoning, not computation.

Larry (28 years old): A social worker with a master's degree in criminology, Larry decided to get involved with rehabilitation of juvenile delinquents. Part of his new job required him to appear in court on behalf of the young offenders, and to read aloud the report of psychiatrists concerning the delinquents. Larry found that he was unable to phrase his oral reading correctly; his stress and accent of multisyllabic words made his oral contributions almost unintelligible. He understood what he read; indeed, to get his master's degree he doubtlessly read hundreds of similarly worded reports, yet transforming the written word to its oral counterpart proved enormously difficult.

Of course, language difficulties and arithmetic difficulties can result from a variety of causes--including physical handicaps, poor motivation, poor instruction, or general limited intellectual capacity to learn. But, in the cases of Johnny, Bill, and Larry, the difficulties were judged to have resulted from central nervous system dysfunctions. They also exhibited a discrepancy between potential and achievement.

Johnny, Bill, and Larry each illustrate common types of learning disabilities. Nonetheless, neither Johnny nor Bill nor Larry can be considered "typical" LD subjects. In fact, there may be no such thing as a "typical" LD subject. The common element among the learning disabled is their "information processing dysfunction," which may range from severe disturbances of perception to highly specific conceptual deficits. In addition, the way the processing dysfunction becomes manifest depends on environmental situations, individual goals and aspirations, and cultural demands (Wallace & McLoughlin, 1975). Consequently, many qualitatively different types of subjects are lumped together under the category of "learning disabled." At the present time, depending on the definition employed, learning disabled individuals may comprise 1% to 20% of the population.

## The Definition of LD

In an effort to clarify the concept of "learning disabilities" for the special educator, an institute for advanced study was held at Northwestern University (Myklebust & Boshes, 1969). The definition evolving from that institute involved the following three criteria: (a) The Exclusion Clause says that the learning difficulty is not primarily the result of poor instruction or lack of opportunities to learn, nor of sensory, motor, intellectual, or emotional handicaps--e.g., poor physical health, motor problems such as cerebral palsy, blindness, deafness, general mental retardation. (b) The Discrepancy Clause says that the child should demonstrate a discrepancy between actual and expected achievement in one or more areas--e.g., auditory receptive language, auditory expressive language, reading, writing, mathematics and non-verbal areas including time orientation, left-right orientation, spatial orientation, body image, social perception, perceptual-motor, picture interpretation, and self-help skills. (c) The Central Processing Deficits Clause says that learning difficulties should stem from central processing deficits--e.g., neurological or psychoneurological dysfunctions involving (for example) perception, integration, or expression.

Since the 1969 institute, different restatements of definition have been offered. Nonetheless, most contain the three basic criteria, either explicitly or implicitly (National Advisory Committee on Handicapped Children, 1968). For example, as recently as November, 1975, the following restatement of definition was proposed (Department for Children with Learning Disabilities, Note 1):

> A specific learning disability is a serious impediment to cognitive functioning which:
>
> (a) is manifested in such wide discrepancies among developmental and/or school achievement areas that special, remedial and/or compensatory teaching is required;
>
> (b) exists independently of or in addition to mental retardation, sensory deficits, emotional disturbance, or lack of opportunity to learn.

The term "cognitive functioning" corresponds to "central processes" in the 1969 version, and the "discrepancy clause" and "exclusion clause" appear almost unchanged in the "new" 1975 version.

Even though recent refinements of the definition seem quite precise in a formal sort of way, ambiguities often arise in practice because of weaknesses in the accepted diagnostic instruments that are available. Senf (1977) states, "If we cannot define those children who represent the kernel of our concern, then we shall never generate an informational base concerning their problems." Operational definitions are needed to explain key elements of the criteria, such as "cognitive functioning," "remedial teaching," "wide discrepancy" and so forth (Department for Children with Learning Disabilities, Note 1). However, from the point of view of the present article, concern about definitions of LD subjects is of interest primarily because it involves an attempt to give precise behavioral descriptions of abilities that are deficient in LD children. Marolda (1977) writes: "The study of specific learning disabilities in mathematics is dependent upon our ability to define and assess mathematical ability." Some examples are given below to illustrate difficulties related to ambiguities in the definition of mathematical abilities.

## The Central Processing Clause

With respect to the central processing deficits clause, many psychoneurological dysfunctions (e.g., spatialization) are not easy to isolate and diagnose. Furthermore, connections between neurological dysfunctions and specific conceptual abilities have not usually been established. Consequently, if it is impossible to correct the neurological or psychoneurological dysfunctions, it is difficult to know how the affected abilities can be improved or refined. That is, it is difficult to find compensating strategies that might be helpful to people with particular types of disabilities. Effective remedial activities are difficult to prescribe until more is known about relationships between central processing dysfunctions and important underlying mathematical abilities. But this will require an analysis of causal connections and precise description of the mathematical abilities that are most often affected.

## The Exclusion Clause

The exclusion clause presents difficulties because of the "chicken-egg" causal relationship between central processing difficulties and other factors associated with learning difficulties--e.g., poor motivation, low tolerance for frustration, poor peer group relations, poor health, poor judgment in social and interpersonal situations. Often, especially with older children or adolescents, it is difficult to determine whether a youngster's learning difficulties have been caused by social or emotional difficulties, or whether the social and emotional difficulties were caused by an apparent disability.

It is difficult to eliminate the possibility that an apparent learning disability results from poor instruction, lack of opportunities to learn, lack of motivation, or general mental retardation. Consequently, regardless of what formal definition is accepted to describe LD subjects, the fact remains that many youngsters who are assigned to LD classes frequently do not really conform to the intended definition. Once again, too little is known about the nature of the abilities that are affected by various neurological, psychoneurological, physical, social, or emotional factors. Therefore, too little is known about relationships between these factors and specific learning disabilities.

## The Discrepancy Clause

The discrepancy clause is perhaps the most useful criterion for identifying LD subjects. Because it is the criterion most closely linked to specific abilities that are lacking in individual subjects, it is also the most useful for helping to prescribe appropriate remedial activities. The frames of reference used in designing the diagnostic battery can (or should) serve as guides to the selection of appropriate remedial activities.

There are two general and closely related ways the discrepancy clause can be satisfied--one is a horizontal discrepancy and the other is a vertical discrepancy. A vertical discrepancy occurs when the actual achievement lags behind expected achievement in a given area of concept formation.* A horizontal discrepancy occurs when a child's abilities in one area of concept formation are "out of phase" with abilities in other areas. For example, test scores on the Wechsler Intelligence Scale for Children (Wechsler, 1974) may indicate characteristic types of "scatter" involving high verbal-low performance or low verbal-high performance scores.

As the above descriptions indicate, both vertical and horizontal discrepancies depend heavily on the effectiveness of standardized testing instruments to isolate specific cognitive abilities. But, "abilities" are difficult to measure using standardized testing instruments. So, "achievement" is typically used as an indirect measure of ability. Unfortunately, however, ability and achievement are not equivalent. For example, Krutetskii (1976), gives examples from his research to illustrate that "The same progress in different pupils can be an index of different abilities, and those with identical abilities can differ in their progress." Krutetskii argues that tests show which tasks can and cannot be done by a child, but they do not disclose how a child has arrived at a practical solution of a certain problem; they say nothing about the reasons for lack of success.

> ...In most cases where the same test result has been obtained the mental processes that have led to the result can be essentially different. And this very difference can be the most valuable material for judging an examinee's psychological traits--his abilities.... (p. 14)

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* Expectancy age (EA) can be calculated as a weighted average (for example, see equation 1) of several other factors including chronological age (CA), grade age (GA), and mental age (MA)--where mental age is determined on the basis of IQ scores or some other measure of general intelligence or achievement. A learning quotient (LQ) can then be determined for a particular category of abilities (e.g., mathematics) by calculating the ratio of actual achievement to expected achievement in the given area (for example, see equation 2).

Equation 1

$$EA = 1/3\ (CA + GA + MA)$$

Equation 2

$$LQ = \frac{AA}{EA} \times 100$$

In the past, learning disabilities subjects have sometimes been defined as youngsters whose IQ's are normal or above normal (e.g., IQ = 90) but whose learning quotients are below a certain cutoff point (e.g., LQ = 89) in a given area.

## Using Standardized Tests to Identify Mathematical Disabilities

Standardized tests have played a central role in attempts to identify both mathematical abilities and disabilities. But the use of standardized tests poses many difficulties for abilities research.

> The reform of the school mathematics curriculum in the United States over the last two decades has not been accompanied by a comparable reform in mathematics testing. Most standardized mathematics tests in use today have undergone modest revision at best, and teachers have cast about in vain for ways to measure the "higher cognitive abilities" that the new curriculum claimed to develop. There has been no test development effort, no sustained research program, and no statement of underlying theory that mathematics educators could turn to as a basis for understanding what these higher abilities might be, let alone how they might be developed. (Kilpatrick & Wirszup, 1976, p. xv)

Furthermore,

> ...Because of the emphasis on correct solutions within fixed time periods, these tests do not help in determining the reasons for errors, nor in discovering the point at which learning has broken down. Another limitation of these instruments is that they do not allow consideration of varied thinking or learning styles. (Marolda, 1977, p. 589)

One of the major difficulties with using currently available tests to identify learning disabilities is that the tests were designed to find children who are having difficulties (or who excel) rather than to identify the specific difficulties of an individual youngster.

If the function of a test is simply to identify children who are having difficulty, rather than to isolate the particular difficulty of an individual child, then it may not be important to determine why individual items are missed, or to investigate how conclusions are found for specific questions. Nonetheless, Krutetskii's research shows that focusing on specific errors of individual children, and "following their thinking" to determine how conclusions are formed is crucial for a thorough analysis of mathematical abilities. Because the same solution to a mathematical problem can often be obtained in quite different ways, Krutetskii argues that one cannot discover much about thinking by analyzing only final responses on standardized test items. Often, the hallmark of outstanding problem solvers is not so much whether answers to specific questions are right or wrong but whether "clever" procedures were used in the attempt. "Good" problem solvers are typically flexible thinkers who are capable of solving problems in several different ways; so, when one path is blocked, another route can be taken. Quality, as well as quantity of answers, must be considered. For example, speed alone can be a misleading measure of problem solving ability. Because of the impulsiveness or inflexibility of their thinking, LD youngsters may actually solve some levels of problems (if they are able to solve them at all) faster than their normal ability peers. Yet, impulsivity and inflexibility are common hindrances to general problem solving ability.

Krutetskii also argues that short term tests typically ignore the influences of numerous factors (especially personal factors like motivation, interest, anxiety, and fatigue, but also nonability factors like specific prior training) that affect performance. Krutetskii criticizes "bourgeois" psychologists for assuming that abilities are stably functioning factors throughout a person's lifetime and that a person's profile of abilities is relatively unaffected by experience and instruction. The "ability categories" (e.g., "perceptual" abilities, "spatial" abilities, "associative memory" abilities, etc.) that most standardized testing procedures identify seldom specify processes that learning specialists can modify through instruction. Yet, the kinds of abilities that will be most useful to LD specialists are those that can be influenced by instruction.

## The Uniqueness of Mathematical Abilities

Do exceptional children (either "gifted" or "learning disabled") really use qualitatively different processing abilities and problem-solving strategies from "normal" children; or are differences primarily due to motivation, or knowledge of specific prerequisites in an isolated subject matter area? The discrepancy clause is based on the idea that the abilities in one area (e.g., mathematics) are either (a) "out of phase" with abilities in other areas or (b) inconsistent with some measure of general ability. However, the existence of cognitive abilities that are unique and specific to mathematics and not simply attributable to "lack of knowledge concerning prerequisite facts" is by no means universally accepted among psychologists, mathematicians, or educators. For example, distinctions between mathematical algorithms and grammatical systems do not seem to be as clear as many people have naively assumed.

> Understanding a paragraph is like solving a problem in mathematics. It consists in selecting the right elements of the situation and putting them together in the right relations, and also with the right amount of weight or influence or force for each. The mind is assailed as it were by every word in the paragraph. It must select, repress, soften, emphasize, correlate, and organize, all under the influence of the right mental set or purpose or demand. (Thorndike, 1917, p. 326)

Many aspects of language development implicitly involve the mastery of certain logical-mathematical rules and systems (Sinclair-de-Zwart, 1969) and significant portions of elementary mathematics involve little more than the mastery of specialized verbal-syntactical systems (Beilin, 1975, 1976; Chao, 1968; Werner & Kaplan, 1963). The processes and abilities required for two mathematics tasks may be quite different, whereas a mathematics task and a language task may involve nearly identical underlying processes. Under these circumstances, labeling a given process as "mathematical" or "nonmathematical" can be quite artificial. Future research on mathematical disabilities should involve careful analyses of the processes that are actually needed to perform specific tasks.

Although a number of studies have claimed to furnish evidence that ability in mathematics is distinct from general intelligence, the influence of non-ability factors like "motivation and interest," "prior training or practice in specific content areas," and "lack of specific bits of information or specific prerequisite skills," have usually been ignored. For example, Gagné's research (1970) indicates that, after differences in prerequisite knowledge have been factored out, mathematical abilities may be nothing more than specific manifestations of general intelligence.

Several mathematicians (e.g., Poincare, Hadamard, Kolmogorov) and psychologists (e.g., Krutetskii, Binet, Revesz) have argued convincingly for the uniqueness of mathematical abilities. Nonetheless, these individuals have been careful to distinguish between ordinary "school" ability and independent, creative mathematical ability. For example, Kolmogorov (cited in Krutetskii, 1976) stated that "ordinary, average human abilities are quite sufficient for mastering--with good guidance or good books...--the mathematics that is taught in secondary school" (p. 4). Krutetskii writes, "Anyone can become an ordinary mathematician; (but) one must be born an outstanding, talented mathematician (1976, p. 361)."

The above opinions indicate that if researchers study only normal "average ability" youngsters, and if they investigate only low level concepts (the kinds tested in most standardized tests), it may be very difficult to identify abilities that are unique to mathematics. This point of view is strengthened by the fact that evidence supporting the existence of special mathematical abilities is most available from research involving gifted students. On the other hand, in later sections of this paper it will be argued that, by neglecting to work with learning disabilities students or other students whose mathematical abilities may differ from those of normal children, researchers may have overlooked certain elementary but critically important mathematical processes.

Little work has been done to isolate and describe specific mathematical disabilities of LD youngsters. LD specialists have tended to study only the lowest levels of mathematical abilities and concepts (e.g., simple calculation skills) and have neglected a broad range of important underlying concepts (e.g., regrouping) and processes (e.g., modeling). Furthermore, the real mathematical status is questionable for many of the concepts (e.g., number conservation, seriation) which have been investigated, and it is unclear how some "abilities" which have been studied (e.g., imagery, tactile-kinesthetic or motor skills) are related to standard ideas that typically occur in an elementary school curriculum (Lesh & Mierkiewicz, 1978).

## Summary

It seems unlikely that mathematical disabilities will be isolated unless tasks are presented that involve real mathematical substance, and unless mathematically skilled interviewers conduct clinical interviews to focus on the specific mathematical processes used by individual children. Unfortunately, the cross disciplinary expertise required in this type of research, and the time consuming nature of the techniques used pose significant obstacles for this type of research. One of the most obvious difficulties for anyone attempting to bridge the theoretical gap between learning disabilities and mathematics education is a general lack of communication between these two fields. For example, many of the words that mathematics educators use to describe the acquisition of mathematical concepts (e.g., isomorphic systems, concrete embodiments, modeling processes) are unfamiliar to most learning disabilities specialists. Similarly, ideas such as "semi-autonomous systems," "preferred modality," and "overloading" are unfamiliar to mathematics educators. Nonetheless, many ideas and techniques from learning disabilities (e.g., clinical teaching cycle, frames of reference) appear to be potentially useful to mathematics educators, and many of the ideas and instructional materials from mathematics education (e.g., mathematics laboratory activities using cuisenaire rods, arithmetic blocks, geoboards, counting discs) appear to offer potentially useful ideas for learning disabilities specialists. To highlight some of these potentially useful ideas, several theories familiar to mathematics educators will be used in the remainder of this paper to describe some abilities that LD youngsters may lack. Hopefully, by focusing these theories on LD subjects, some new insights may arise concerning abilities normal children must refine to master mathematical concepts--and, some of the abilities identified will be helpful to LD specialists who must conduct diagnostic interviews and prescribe remedial activities for children with learning difficulties in mathematics.

## II. Abilities Associated With Gagné's Theory of Learning

This section will emphasize three categories of abilities: (a) abilities associated with adjusting learning or problem-solving styles to fit different task situations, (b) abilities associated with information storage and retrieval, and (c) abilities associated with lower order forms of learning.

### The Ability to Adjust Learning Styles to Fit the Task

One of the fundamental principles underlying Gagné's (1970) theory is that there are different varieties of learning that can be distinguished by means of the conditions that produce them. For example, after analyzing the things that an instructor must do to produce various types of learning, Gagné identified eight varieties of learning: (1) problem solving--i.e., combining rules into higher order rules, (2) rule learning--i.e., learning relations between lower order concepts (or rules), (3) concept learning--i.e., making a common response to different stimuli, (4) discrimination learning--i.e., making a different response to different stimuli (5) verbal associations or verbal chains, (6) stimulus-response chains, (7) simple stimulus-response learning, and (8) signal learning.

A basic assumption underlying Gagné's scheme is that a teacher, instructional program, or student should do different things (i.e., provide different conditions) to produce each of the eight types of learning. Conversely, if a teacher provides the same kind of learning activities every day, the same type of learning can be expected to occur--whether or not the instructor intentionally planned for this to be the case.

> Maurice (fifth grade): Maurice is quite verbal. He can state a correct rule about most of the ideas in his mathematics book. For example, when asked about finding the area of a triangle, he stated that the rule was to "multiply one-half of the base times the altitude." Unfortunately, Maurice's performance in mathematics had been very poor. Although Maurice seemed to have mastered an unusually large number of mathematical rules, most of what he knew were not rules at all according to Gagné's criteria. Maurice was unable to use his rules in most simple concrete situations and he had difficulty describing what most rules meant in situations involving simple pictures or instructional materials. According to Gagné's classification scheme, most of Maurice's "rules" were merely verbal chains.

Normal children (at least those who are doing well in mathematics) have usually learned to provide for their own conditions for learning mathematics--even when their teachers or textbooks neglect to perform these functions. A distinguishing ability of good students is that they do different things to learn different kinds of ideas, whereas learners who are having difficulty are often much less flexible in adjusting their learning styles to tasks they confront. For example, when left to his own devices, Maurice tried to learn most things as though they were simple verbal chains. Furthermore, Maurice's teacher also tended to teach most ideas as though they were verbal chains. That is, the conditions for learning that she provided usually were conditions that facilitate the learning of verbal chains--not rules or concepts. Consequently, the teaching style of his teacher was compounding Maurice's difficulties rather than helping him overcome them.

Lack of flexibility in adjusting his learning or problem-solving techniques to fit specific tasks was not a difficulty unique to Maurice. For example, Maurice's class had been learning to use a problem-solving strategy that his teacher called "make a graph." Many of the children became quite skillful at making graphs when they were told to do so, but no time was spent helping the children learn to identify situations where this particular strategy might be appropriate. Therefore, one week after the "making graphs" unit had been completed, none of the children--not even the exceptionally bright students--in Maurice's class used the "make a graph" strategy to try to solve problems in which Maurice's teacher thought it would "obviously be helpful."

Getting an idea into a youngster's head does not guarantee that the new idea will be integrated with other ideas that are already understood; it does not guarantee the youngster will be able to use the library-type "look up" skills that allow the idea to be retrieved when it is needed; nor does it guarantee that he will be able to identify situations in which the idea is useful or relevant. For example, for Maurice, knowing how to carry out pencil and paper computations did not ensure that he knew when to compute, which operation to use in simple everyday problems, or how to use the answers once they were found. Similarly, other children in Maurice's class had learned a variety of problem solving strategies or heuristics, but they had not learned: (a) to identify situations in which particular strategies might be useful, (b) to identify stages in the problem solving process when particular strategies might be useful, (c) relationships among various ideas and strategies.

There was no evidence to prove that Maurice's tendency to learn most rules as though they were verbal chains resulted from some underlying learning disability. But, he had been enrolled in "help sessions" with his school's LD specialist, who, after extensive testing, was still unable to identify the cause of Maurice's learning difficulties. Did they result from an underlying learning disorder or had he simply failed to learn some basic ideas that caused particular difficulties in mathematics? In either case, the diagnostic and prescriptive teaching techniques that the LD teacher used were very helpful to Maurice. At the end of the year, he was doing acceptable work in mathematics and he was far more flexible in adjusting his learning style to particular task situations.

Because many different types of rules, laws, tasks, and ideas are encountered in school, flexible adjustment of learning style is important for all students. For example, the inductive strategies required to find the $n^{th}$ term of a given number sequence are quite different from the deductive strategies needed to prove a trigonometric identity. The repetition and reliance on cues that characterize learning important names and dates in history would not help the student analyze the causes of the Civil War. Difficulties can arise if the student is unable to adapt his learning or problem-solving procedures to fit the conditions for learning in different situations.

## Abilities Associated With Information Storage and Retrieval

Gagné analyzes the act of learning into separate "information processing" phases: (a) an apprehending phase, (b) an acquisition phase, (c) a storage phase, and (d) a retrieval phase. He also partitions these phases into smaller categories. For example, the storage phase is partitioned into (i) temporary holding, (ii) mediational holding, and (iii) lifetime retention, and the retrieval phase is partitioned into such categories as recognition memory and reproductive memory.

In mathematics, as in most subjects, some facts are only worth remembering for a short period of time (as when someone tells you a telephone number); some facts are only for mediational use (they contribute to learning a higher order rule but may be forgotten after the higher order rule has been learned); and some facts are appropriate candidates for lifetime retention. However, in mathematics, relatively few facts fall into the latter category. Most isolated facts can be regenerated from other facts if they are forgotten. The more regeneration rules a child learns, the easier it is to remember those basic facts that are most important. Consequently, instead of learning to recall isolated bits of information, outstanding mathematics students usually focus on remembering broad categories of information that will efficiently organize many details and serve as cues for mentally "looking it up." On the other hand, students who are having difficulty in mathematics, commonly lack the flexibility of thought that allows them to switch quickly to appropriate types of memory functions, and they often lack the ability or inclination to organize information into broad categories so they can effectively retrieve information relevant to a given problem or task.

The preceding section described Maurice's tendency to memorize all facts as isolated bits of information. He implicitly seemed to consider all facts to be equally important and worthy of being memorized forever. Therefore, even though Maurice had a remarkable ability to remember isolated facts, information overload eventually caused his memory to fail. Perhaps some of these retrieval skills correspond to "abilities" that LD youngsters may lack, but it seems likely that inferior retrieval skills often result from the general disorganized nature of a body of knowledge--plus a lack of flexibility in adjusting learning and memory factors to different task situations.

Efficient information retrieval systems usually require well organized systems of knowledge. To be remembered when they are needed, new ideas must be cross-referenced using flexible organizational schemes that tell how individual ideas are related to other ideas. One of the characteristics of gifted students is their tendency to compare new ideas with a vast range of things they already know (Davis, Jockusch, & McKnight, 1978), and one striking characteristic of many students who experience difficulty in mathematics is the disorganized nature of the mathematical information they have learned.

> Kathy (fourth grade): Kathy had mastered a large quantity of verbal information about mathematical ideas. Yet she was doing very poorly in her mathematics classes and on math related sections of most tests. Even though Kathy had memorized all of her addition and multiplication facts, as well as most subtraction and division facts, there was little evidence that she had organized these facts. Also Kathy was usually unable to use facts that she knew as substeps in more complex problems. She was unaware of most of the number patterns (e.g., multiples of five, multiples of nine) and properties (e.g., 3 X 5 = 15 and 5 X 3 = 15) that most children use to organize the arithmetic factors they learn, and she seemed unaware of the relationship between addition facts like 7 + 5 = 12 and subtraction facts like

12 - 5 = 7. Remarkably, Kathy had succeeded in memorizing nearly all subtraction and division facts without relating them to addition and multiplication.

Mary (sixth grade): Mary had difficulties with most kinds of computation problems. Surprisingly, however, she had become quite skillful at calculating multiplication problems like 38 X 27 = □. Yet, she had extraordinary difficulty performing "regrouping" addition problems like 266 + 760 = □ even though, for most people, the addition problem 266 + 760 = □ is a substep in the multiplication problem 38 X 27 = □. Among Mary's difficulties was the fact that she saw little connection between "addition in an addition context" and "addition in a multiplication context."

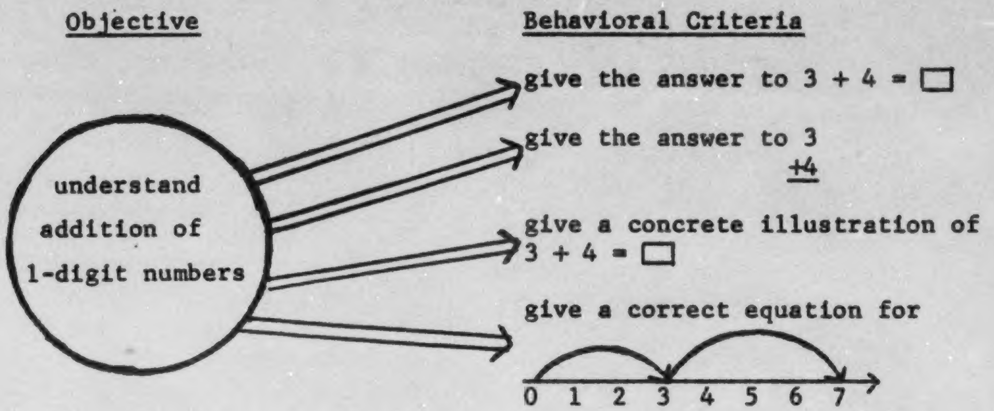
Ideas that are logically related (e.g., 9 X 7 = 63 and 63 ÷ 7 = 9) may not be psychologically related in the minds of students. Just because adults consider two situations to be "alike" does not mean that children will consider them to be similar. A distinguishing characteristic of gifted children is that they have an ability to recognize similarities among seemingly unrelated situations; and a distinguishing characteristic of many LD children is that situations which educators, mathematicians, and psychologists usually consider to be closely related may be treated by an LD child as being quite unrelated.

Mathematics, probably more than any other subject matter area, is based on well organized systems of relationships among ideas and skills. Consequently, a disability affecting the overall organization and structure of a child's concepts could be expected to have a greater detrimental effect on mathematics than on other subject areas.

Problems associated with fragmented knowledge systems are quite common among students at all age levels and in all types of mathematical tasks. For example, in his research involving the use of problem-solving strategies by college students, Schoenfeld (in this volume) demonstrated that learning managerial strategies for a system of specific strategies was quite different from simply learning each of the strategies in isolation. Similarly, fragmentation also occurs among younger children in simpler learning situations. For example, teachers or researchers dealing with children who have learning difficulties tend to break down complex tasks and ideas into sequences of smaller discrete learning units. This process of analysis can be quite useful, but it can also reinforce the fragmented nature of children's thinking. Teachers must do more than teach isolated sequences of ideas. Otherwise, their analytic teaching procedures may actually accentuate the very things that are causing difficulties for the child--producing short range success which lays the groundwork for even greater future difficulties. In mathematics, a whole system of ideas is often more than the sum of its parts; and to teach or conduct research as though this were not true demonstrates a misunderstanding of the nature of mathematics--and the thought processes children use to do mathematics.

Lesh

Behavioral objectives furnish another example of an instructional component that can produce fragmentation of content if they are misused. According to Gagné's theory, an instructional unit should be accompanied by behavioral criteria for determining whether the objectives of the unit were achieved. For example, if a teacher's objective concerned adding one digit numbers, some of the behavioral criteria might be: (a) the child can give (write, name, or select) the answer to equations like $3 + 4 = \square$, (b) the child can give (write, name, select) the answer to equations like $\begin{array}{r}3\\ \underline{+4}\end{array}$, (c) the child can use concrete materials (i.e., poker chips, Cuisenaire rods, a number line) to illustrate problems like $3 + 4 = \square$, (d) the child can write (or describe verbally, or select from a set of choices) the correct equation to describe a given concrete illustration of addition.


Objective
Behavioral Criteria
understand addition of 1-digit numbers
give the answer to 3 + 4 = ☐
give the answer to 3 +4
give a concrete illustration of 3 + 4 = ☐
give a correct equation for
0 1 2 3 4 5 6 7

Sometimes "behavioral criteria" are called "behavioral objectives"--implying that the objective of a unit is simply to get children to behave in a particular way on particular tasks. Nonetheless, it is usually possible to teach students to perform the designated behaviors without having learned the underlying objective. For example, the student may be able to perform each of the isolated behavioral objectives but may not understand how they are related to one another.

> Johnny (second grade): Johnny could give correct answers to most addition problems of the form $\begin{array}{r}7\\ \underline{+8}\end{array}$, and to many problems of the form $7 + 8 = \square$. Surprisingly, however, the systematic errors he made on horizontal addition problems did not necessarily correspond to his errors on vertical problems. He seemed to treat the vertical and horizontal forms of addition problems as two unrelated systems. He could illustrate problems like $3 + 4 = \square$ using either poker chips or a number line, but he had great difficulty translating directly from a number line illustration to a poker chip illustration (or vice versa). Again, he seemed to treat poker chip illustrations and

number line illustrations as two unrelated systems.

The following two teaching objectives are quite different: (a) teach a child to understand an idea in such a way that he will behave in a particular way on a given task, (b) teach the child to perform a given task. The distinction between these two types of objectives becomes especially apparent in research on "systematic errors" (e.g., Ashlock, 1972; Cox, 1975). In many cases, and particularly in cases involving LD subjects, the errors children make are quite systematic and are based on identifiable rules. In most cases, the incorrect rules were able to generate correct answers for a small class of problems, but are unable to generate correct answers for a larger class of problems.

Jennifer (fourth grade): In a series of subtraction problems, she gave the following responses.

$$\begin{array}{rrrrrr} 47 & 52 & 68 & 74 & 82 & 56 \\ \underline{-24} & \underline{-37} & \underline{-35} & \underline{-28} & \underline{-57} & \underline{-44} \\ 23 & 25 & 33 & 54 & 35 & 12 \end{array}$$

Clearly, Jennifer's difficulties did not result from random errors. She had learned a rule that worked for subtraction problems that did not involve regrouping, but her rule gave incorrect answers to regrouping subtraction problems.

Ashlock's book, Error Patterns in Computation (1972), is filled with examples of systematic errors that are quite common among both LD and non-LD students. But, an unusual proportion of LD children seem to generate systematic procedures which (for a short period of time involving some small class of problems) give right answers for the wrong reasons.

In the same way that a computer can usually use many different programs to produce "correct answers" on a given set of problems, many children are also quite creative in their abilities to create non-standard procedures for performing specific tasks. Unfortunately, many of these non-standard techniques will only yield correct answers for a restricted class of problems. The teacher's objective is not simply to modify a child's overt responses on a given task, she must also modify the "internal programs" the child uses to generate the responses. Most instructional situations can be expected to have both behavioral criteria and cognitive objectives.

## Abilities Associated with Prerequisite Forms of Learning

According to Gagné, one of the most important factors influencing a youngster's ability to learn a new idea is the extent to which prerequisite ideas and skills are available. Starting with a clear statement of the idea one wants to teach, it is possible to derive a "learning hierarchy" of prerequisite ideas by progressively asking "What would a student have to know (or be able to do) in order to know (or be able to do) that?" By asking this question, first for the terminal objective, and then for each of the prerequisites, an upside-down tree diagram (i.e., a "learning hierarchy", (see Figure 1) can be constructed. The final objective can be visualized as the trunk of the tree with prerequisite concepts on the branches below.

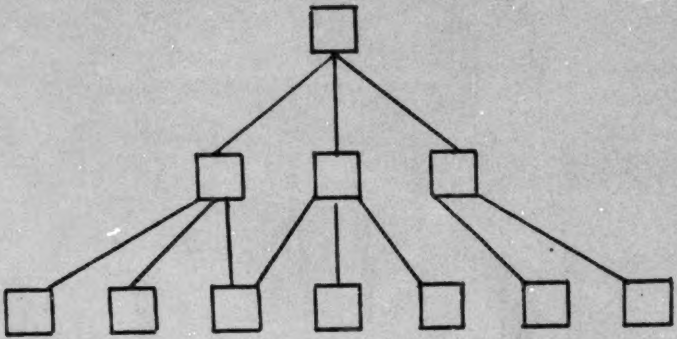


Figure 1. A tree diagram.

Another important feature of Gagné's theory is that the eight basic types of learning are ordered in a prerequisite sense. That is, if the objective of an instructional unit is to learn a rule, then the prerequisites will be concepts and/or lower order rules. The prerequisites for concepts are discriminations; and the prerequisites for discriminations are among the lower types of learning. This ordering of dependency relationships is summarized in Figure 2.

Problem Solving
↘
Rules
↘
Concepts
↘
Discriminations
↘
Verbal Chains
↘
S-R Chains
↘
Stimulus Response Learning
↘
Signal Learning

Figure 2. Dependency relationships in Gagné's basic types of learning.

According to Gagné, one of the most important functions a teacher should perform is to begin instruction with an evaluation of each student's initial state of learning. A youngster cannot be expected to learn a rule if the prerequisite rules and concepts are not available; and the prerequisite concepts cannot be mastered if the youngster is unable to make the appropriate discriminations.

> Andy (fourth grade): Andy is extremely bright and is an excellent problem solver. Nonetheless, he has great difficulty with reading and arithmetic. In arithmetic, even simple one-step subtraction problems cause difficulties for him. He memorized his addition and multiplication tables perfectly, but he commonly misses subtraction problems like 73 - 5 = ☐ . Andy's answer to this question was 23; that is, he subtracted the 5 from the 7 instead of the 3.

For most children who would give answers like the one above, a teacher would probably assume that the difficulty resulted from a lack of understanding of place value (i.e., 73 stands for 7 tens and 3 ones) or from a systematic error similar to Jennifer's in the preceding section. Actually, Andy seemed to have a learning problem that was even more basic than a systematic error involving place value. He had difficulty discriminating 73 from 37. He also had "figure reversal" difficulties when he tried to distinguish "b" from "d" or "p". For example, on multiple choice "shape recognition"tasks, Andy would be just as likely to match ◿ with ◺ as to match ◿ with ◿ .

Andy had been diagnosed as having a central processing dysfunction known as dyslexia.

In Gagnean terminology, Andy cannot be expected to learn the higher order rules (e.g., regrouping) and concepts (e.g., place value) that are relevant to problems like 73 - 5 = ☐ if he has not learned the prerequisite discriminations.

When teaching arithmetic to normal fourth graders, a teacher may simply assume that the relevant discriminations can be made. But LD youngsters may have difficulties with some of these prerequisite skills--e.g., discriminations, verbal chains, S-R chains. Consequently, some of the "disabilities" commonly associated with LD youngsters would be interpreted as "missing prerequisite ideas" according to Gagné's scheme. Whether these missing prerequisites should be called "disabilities," whether they result from even more basic sets of abilities, or whether they are simply unlearned prerequisite ideas is an unsettled question. But a major asset of Gagné's scheme is that it allows LD specialists to "map in" processes and abilities from a number of psychological and educational theories, and to organize them into a single framework. It also allows processes, abilities, or ideas at one level to be related to processes, abilities, or ideas at other levels. For this reason, Gagné's theory could furnish a valuable system to use to search for abilities LD children may lack.

## Summary

Three types of abilities were discussed in this section: (a) abilities involving adjusting learning or problem-solving styles to fit different kinds of task situations; (b) abilities associated with information storage and retrieval; and (c) abilities associated with prerequisite forms of learning. Deficiencies in these abilities need not prevent children from engaging in meaningful problem-solving situations. Gagné's book Conditions of Learning (1970) describes conditions for problem solving that are accessible to many LD children with mathematical learning disabilities like the ones described throughout this paper. Furthermore, Gagné's theory can also help teachers or researchers coordinate useful information from a variety of other theoretical perspectives. For example, Pascual-Leone (1976) has developed a theory concerning the role of memory functions in learning and problem solving. Pascual-Leone's theory fits into Gagné's scheme and it includes specific techniques for anticipating and minimizing difficulties associated with memory functions.

Not only is it possible for LD children to participate in meaningful problem-solving situations, but appropriately designed problem-solving situations can provide excellent instructional settings to help them: (a) develop effective information storage and retrieval abilities; (b) organize the ideas they have learned into flexible systems; and (c) identify situations where they can use the ideas they have learned.

All of the abilities identified in this section to some extent involve Gagné's treatment of "structure." For Gagné, concepts and ideas are elements of a hierarchial structure in which any given idea is related to certain prerequisite ideas. New ideas are "built up" in somewhat the same way that a brick wall is built--by placing each new brick on a solid foundation of bricks at a lower level. However, the "brick wall" model tends to de-emphasize the fact that ideas (unlike bricks) exist at many different levels of sophistication, and that they are not simply "completely mastered" or else "not understood at all." Ideas, just like children, develop through identifiable stages; and careful descriptions of primitive conceptions of an idea can reveal many things about the abilities that are present or absent for children at particular stages in the development. Piaget's theory focuses on precisely this aspect of learning. The next section of this paper will attempt to identify mathematical abilities using a Piagetian perspective quite different from Gagné's.

Gagné's point of view is a potentially valuable resource for investigating some types of mathematical abilities that LD children may lack. Unfortunately, it also ignores or de-emphasizes other types of abilities that seem highly important in mathematics learning. For example, Gagné's theory is obviously more thorough in its description of the lower of his eight types of learning--and it is noticeably weaker in explaining the higher forms of learning (e.g., problem solving, rule learning) that are most prevalent and important in mathematics. In contrast, Piaget has concentrated primarily on these higher forms of learning.

Gagné's theory does discuss certain information processing variables that are useful in problem solving, and it discusses certain abilities that are needed to organize information into categories that will facilitate the retention, retrieval, and flexible use of rules, but Gagné's learner is a fairly passive recipient of knowledge, and the knowledge the learner receives is not characterized as a dynamic and constantly changing system of ideas that must be constructed by the learner. A view of an active rather than a passive learner could suggest additional abilities and/or disabilities. In this light, Piaget's theory will be examined as a source of mathematical abilities.

## III. Abilities Associated with Piaget's Theory of Learning

From a mathematician's point of view, the idea that "mathematics is a verb" expresses a truth that is seldom taken seriously by non-mathematicians. To a mathematician, mathematics is as much a process (i.e., something one does) as it is a product (i.e., something one possesses), The process aspect of mathematics furnishes a likely place to look for mathematical disabilities. But, what is this process aspect of mathematics? In part, mathematical processes include problem-solving strategies or "question asking" techniques and modeling processes that will be discussed in Part IV; but, more importantly for the purposes of this section, mathematical processes involve systems of relations, operations, or transformations that must be coordinated in order for children to make correct mathematical judgments. For example, children in the second grade commonly experience difficulties with two digit addition problems which require carrying (e.g., $27 + 36 = \square$). Such children often do not understand the "regrouping" operations that are necessary for understanding our numeration system. Part of the difficulty is that children are typically expected to apply organized systems of regrouping operations to abstract written symbols before they have had experiences applying these regrouping operations in more concrete situations. Such children often find it helpful to work with the following types of materials: a "counting frame" abacus, bundling sticks, arithmetic blocks, Cuisenaire rods, or unifix cubes.

In two digit addition problems it is clear that a system of operations needs to be coordinated in order for the "regrouping" concept to be understood. According to Piaget, however, most mathematics concepts implicitly require children to master some system of operations or relations (Beth & Piaget, 1966). Unfortunately, however, little is known about the exact nature of the operational and relational structures that children use to make most primitive mathematical judgments. In fact, educators have too often either ignored the operational aspects of mathematical concepts or else they have assumed that the systems of relations children use are identical to those used by adults.

Some of the best resources for describing the nature of children's early mathematics concepts have come from Piagetian studies. Nonetheless, because Piagetian research has focused on the cognitive processes used by first-graders (i.e., concrete operational groupings) and by sixth-graders (i.e., INCR groups), children at intermediate levels of development have been neglected. Furthermore, because psychologists in general (and Piagetian psychologists in particular) have avoided mathematical ideas that are typically taught in elementary school, it is usually possible to make only relatively crude inferences about how children's mathematical thinking gradually changes from concrete operational concepts to formal operational concepts. It is time for mathematics educators and others who are interested in mathematical abilities to apply Piagetian techniques and theory to concepts that exist at intermediate levels of development--as well as at adult or preschool levels or to populations of exceptional children.

Investigating the development of ideas in humans is somewhat different from investigating human development. One of the ingenious aspects of Piaget's theory is that he explicitly confronts the facts that: (a) a given idea can exist at many different levels of sophistication, (b) the evolution of an idea can be traced as it develops in the thinking of learners, and (c) the more primitive conceptions of the idea have seldom been accurately described. The first mathematical judgments youngsters learn to make are highly specialized, closely tied to specific content, and involve restricted and "messy" primitive structures that do not give rise to neat, tidy, elegant mathematical theories. For this reason, mathematicians have not taken the trouble to describe the operational or relational systems children use in their early conceptualizations of most mathematical ideas. An example may help clarify this point.

As this paper was being written, a colleague from the mathematics department brought his 3-year-old son into the author's office. To keep the boy entertained, he was given a box of "Lego" blocks which the child used to build a "fire engine." To select the correct block for a particular purpose, the boy seemed to use some sort of measurement activity. But, three year olds generally do not conserve length on simple Piagetian tasks, so the author was skeptical that the child was really measuring in a true mathematical sense. The child was asked, "How did you know which block would fit here?" He said, "I measured," and he demonstrated by finding another block the same length as the first. Next, not wishing simply to take the boy's word for the fact that he was measuring, the father was asked how his son was able to select the correct block. The father answered cryptically, "He measures." So, we all agreed; he was measuring. But, what was the nature of the boy's measurement concept? As Piagetian theory would predict, he did not realize that the distance from the drinking fountain to the waste basket was the same as the distance from the waste basket to the drinking fountain; and in general, he failed nearly every standard Piaget-type task that the author posed concerning the concept of length. Yet, we all had agreed that the child was measuring length. Clearly, if the boy really did have a concept of length, it was an extremely primitive concept that did not have most of the properties mathematicians usually associate with length concepts. But, what were the properties of the boy's concept, and were these sufficient to justify calling it a concept of length? Perhaps he was simply using measurement words without much real understanding; or worse yet, he could have been using them in a way that encouraged incorrect understanding that could hinder later learning about measurement. Was the child's primitive concept a first step in the direction of an adult conceptualization of measurement, or was it a first step in a wrong direction?

Most of Piaget's mathematics-related research has been directed toward showing that, even at the most primitive levels, mathematics concepts must involve the use of simple systems of operations, relations, and transformations--and that if these systems are not yet coordinated, then the concepts cannot be considered to be a first step in the direction of a correct conceptualization. Children who have not yet coordinated the relevant systems of operations or relations for a given concept are called "preoperational" with respect to that concept. By investigating the operational systems children use to make primitive mathematical judgments, psychologists have discovered many important facts about preoperational abilities and about early operational abilities. Therefore, a promising way to identify possible abilities and disabilities of LD youngsters may be to conduct thorough Piagetian analyses of the systems of operations LD children use to make mathematical judgments. Lesh (1976) and Nelson (in this volume) have given guidelines to conduct these analyses using information about known mathematical systems.

LD children are often described in terms that make them sound very similar to normal preoperational (in a Piagetian sense) children. If LD children really are similar to preoperational children, then this fact could be very helpful in devising appropriate instructional activities for LD children. Compared with what is known about the mathematical abilities of LD children, a great deal is known about qualitative differences between the underlying abilities of operational versus preoperational children. Furthermore, comparing LD children to preoperational children can give a more positive dimension to the search for abilities among LD children. When investigating the cognitive abilities of preoperational children, it is important to describe what a child can do as well as what he cannot do. Concrete operational children are not simply children who cannot use formal operational ideas; they are children who can use concrete operational ideas; and sensorimotor intelligence is not simply the absence of concrete operational intelligence; it is a distinct and viable system of knowledge that carries with it its own consistent rules of logic. Similarly, LD children should not be described in terms that are totally negative--that is, as children who lack certain abilities. Just as in the case of Piaget's research with young children, it is important to describe the abilities LD children do have as well as abilities in which they are deficient.

Comparing LD children with preoperational children may be useful for a second reason. Instructional techniques that are effective with preoperational children may suggest similar techniques to be used with LD children. For example, many "mathematics laboratory" materials (e.g., Cuisenaire rods, arithmetic blocks, counting discs, etc.) have been shown to be particularly effective for preoperational children and in some cases they have also been effective with LD children. On the other hand, even though preoperational and LD children may be similar in some respects, there also seem to be a number of dissimilarities. In clinical teaching situations with LD children, the author has been impressed by the fact that some of his favorite and most effective mathematics laboratory activities were complete failures when they were used with some LD children.

Because most educators and psychologists are not familiar with the kinds of mathematical structures that are necessary to discuss abilities within Piagetian theory, and because these mathematical structures are closely related to useful instructional and diagnostic activities, this section will devote extra attention to several important structural ideas from mathematics, and it will relate these structural ideas to instructional variables used in mathematics laboratory forms of instruction. Abilities (or disabilities) will be discussed as they relate to various instructional variables.

## Mathematics Laboratory Activities

Many mathematics laboratory activities emphasize group problem-solving activities using concrete materials. These activities can be quite effective with many children who have mathematical learning disabilities. Nonetheless, each of these types of situations--i.e., small group sessions, problem-solving sessions, motor activities, and concrete materials--can create problems for LD children. According to Clements (1966), the following characteristics are common among LD children:

Concerning Small Group Sessions:

(1) peer group relations generally poor
(2) overexcitable in normal play with other children
(3) frequently poor judgment in social and inter-personal situations
(4) overly gullible and easily led by others
(5) excessive variation in mood and responsiveness--very sensitive to others, frequent rage reactions and tantrums when crossed

Concerning Communication Skills:

(1) impaired discrimination of auditory stimuli
(2) various categories of aphasia
(3) slow language development

Concerning Problem-Solving Sessions:

(1) low tolerance for frustration, easily fatigued
(2) impulsive-explosive, reckless and uninhibited, impulsive then remorseful
(3) impaired concentration and attention span
(4) impaired ability to make decisions, particularly from many choices
(5) frequent thought preservation

Concerning Motor Activities

(1) hyperkinesis
(2) general awkwardness
(3) distorted concept of body image

Concerning the Use of Concrete Materials:

(1) impaired discrimination of size
(2) impaired discrimination of order (left-right, reversals)
(3) poor spatial orientation
(4) impaired judgment of distances
(5) impaired discrimination of figure-ground
(6) thinking overly concrete, poor ability to abstract (p. 11-13)

The above comments should not be interpreted to mean that concrete activities are not appropriate for LD children. In fact, all of the above difficulties are quite common among normal children for which laboratory activities are usually effective. However, not all types of concrete materials and not all types of activities will be conducive to learning. Critical instructional decisions involve determining (a) which materials will be helpful, and (b) which type of activities will be appropriate. These decisions must be based on a clear understanding of the cognitive abilities (or disabilities) of individual children.

To emphasize the instructional implications of the abilities that will be discussed, the remainder of Part III is divided into the following sections: The Role of Concrete Materials in Mathematics Instruction; The Role of Activity in Mathematics Instruction; The Role of Problem Solving in Mathematics Instruction; and The Role of Small Group Interactions in Mathematics Instruction. The examples that will be used will all involve children who were classified by their school system as "learning disabled." Nonetheless, the examples cited should not be interpreted as "typical" LD children. In fact, as the first section of this paper states, there may be no such child as a typical LD child.

## The Role of Concrete Materials in Mathematics Instruction

According to Piaget (Beth & Piaget, 1966) the characteristic feature of mathematical ideas is that they implicitly require students to use coordinated systems of relations or operations to impose structure on perceived events. Just as a "hidden picture puzzle" must be mentally organized before all of the relevant information can be "read out," Piaget has shown that the mathematical information that adults assume they "read out" of objects arise only after certain organizational systems have been imposed on the environment (Lesh, 1976; Lesh & Mierkiewicz, 1978). Dienes' (1969) "concrete embodiments" furnish examples where figurative models have been used to facilitate the acquisition of mathematical systems. The "best" materials are those in which there is some connection between the structure of the materials and the structure of the concept being learned.

For example, for some instructional materials like arithmetic blocks and Cuisenaire rods, structure seems to be built into the materials; and for other materials like the balance beam (See Figure 3), the structure must be imposed. The ideas that are involved in understanding the principles of a balance scale are far more complex than those involved in understanding basic addition or multiplication facts. It is true that multiplication or addition facts can be used to explain the action of a balance beam. But this does not imply that the balance beam will be useful to explain multiplication or addition. The system of actions involved in a balance beam are not isomorphic to the operational structure underlying simple arithmetic ideas.

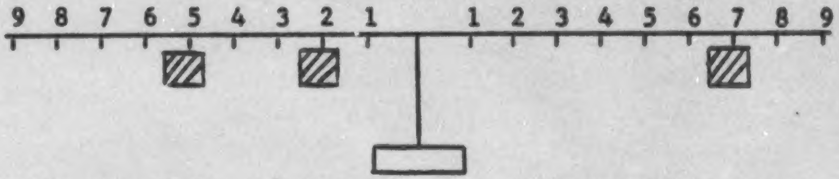


Figure 3. Using a balance beam to illustrate 5 + 2 = 7

In spite of the fact that structure seems to be built into some materials, actually, relational and operational systems must always be imposed on materials. And, the ability to impose structure on concrete materials is sometimes strikingly deficient in some LD children.

> Jimmy (second grade): According to Jimmy's teacher, he was performing well in every subject except mathematics. Actually, he was also a poor writer. His writing seemed more characteristic of kindergartners or first graders. He often reversed or inverted letters (e.g., s,ƨ; m, w) and he had difficulty distinguishing b's from d's or p's. In arithmetic, he seemed to have special difficulty keeping sequences of numbers in order, such as distinguishing 537 from 573. So, his LD teacher decided to use Cuisenaire rods to work with him on ordering ideas.
>
> Jimmy was unable to copy or build a "staircase" of Cuisenaire rods and in general had difficulty recognizing or using the ordering relationships that are so obvious to most second graders. Jimmy had difficulty imposing ordering relationships even in the simplest situations.

What kinds of abilities might be associated with the use of concrete materials in mathematics instruction? Three possibilities include: (1) the ability to impose structure on concrete materials in everyday situations, (2) the ability to translate among various models and interpretations of an idea, and (3) the ability to correctly interpret spatial/geometric aspects of various models for an idea. These three types of abilities will now be discussed.

Imposing Structure on Concrete Materials. In his research with mathematically gifted students, Krutetskii (1976) contends that gifted students have a "mathematical cast of mind," a tendency to interpret the world mathematically. This mathematical cast of mind involves the following three sub-abilities:

- an ability to isolate form from content, to abstract oneself from concrete mater[illegible] and spatial forms, and to focus on the structural properties of problems and situations.

- an ability to generalize from particular situations, to immediately consider particular operations and relations as special instances of more general classes of relations and to focus on what is structurally common among diverse situations.

- an ability to remember the mathematical structure of a problem or situation, disregarding perceptual characteristics or other mathematically irrelevant properties.

Concerning some possible disabilities among poor students, Krutetskii states:

> Analogously, inability in mathematics (also with extreme cases in mind) is caused originally by the brain's great difficulty in isolating stimuli of the type of mathematical generalized relationships, functional dependencies, and numerical abstractions and symbols, and by difficulty in operations with them. In other words, some persons have inborn characteristics in the structure and functional features of their brains which are extremely favorable (or quite unfavorable) to the development of mathematical abilities. (1976, p. 261)

The inborn character of the abilities described by Krutetskii seems questionable. There do seem to be disabilities corresponding to each of the three abilities listed above, but these disabilities seem to be linked more closely to the development of specific structural schemes than to some measure of general mathematical giftedness.

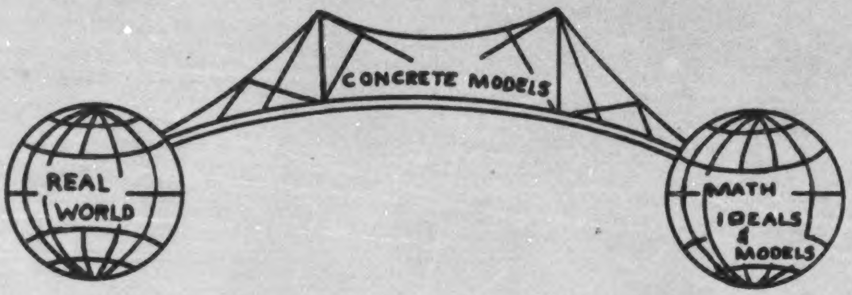
Several recent studies (Chartoff, 1976; Lesh & Mierkiewics, Note 2; Silver, Note 3) have investigated Krutetskii's hypotheses that (a) students of high mathematical ability tend to recognize and remember the mathematical structure of problems and situations, whereas (b) students of low mathematical ability tend to focus on non-structural properties of problem situations. These studies involved 6- to 14-year old students from various ability levels and used multidimensional scaling techniques in addition to more theoretical analyses to identify the kinds of properties students use to evaluate similarities among problems. The problem situations were designed so that some pairs of problems involved the same structure but different content while other pairs involved the same materials but different structures. The results demonstrated that:

> (i) The tendency to recognize and remember the structure of a problem depends first and foremost on whether the relevant system of mathematical relations and operations are known and accessible to the students. For example, noticing the structure of a problem was more closely related to whether or not a student could solve the problem than to the general problem-solving capabilities of the student. That is, a young but highly gifted problem solver might

fail to notice the structure of a given problem--perhaps because the relevant structural relationships were not available; whereas a mediocre problem solver (perhaps older) might attend to the structure of the problem. Overall, for explaining the tendency to focus on problem structure, general problem solving ability did not seem as important as the accessibility of relevant organizational structures.

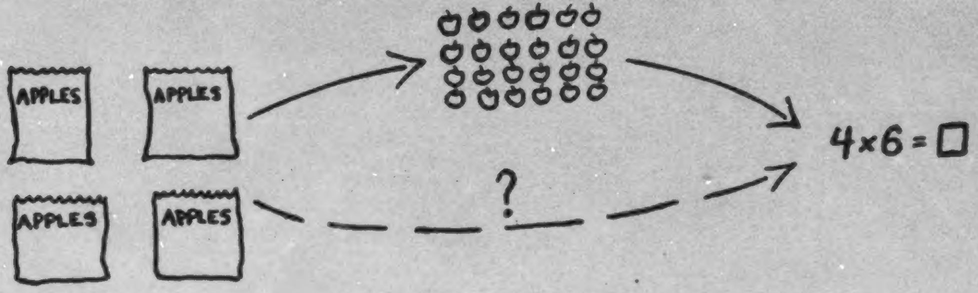
(ii) If all other things were equal, including the availability of relevant organizational structures, more able students *did* have a greater tendency to recognize and remember the structure of problem situations. That is, lower ability students often seemed to *have* the relevant structures, but were unable to use them in given situations.

What is it, beyond having an idea (or organizational system), that enables a student to *use* it in a given situation? Assume that a child is able to impose a given operational system in the simplest or most obvious situations, and then ask what abilities might be needed to extend this operative ability to more complex and less obvious situations. The best concrete embodiments were designed precisely to help children make this transition to progressively more complex situations, and the materials themselves furnish some of the simplest and most obvious situations in which mathematical structures can be used by young children.



Concrete materials can be used as a "bridge" to help children relate their mathematical ideas to specific problems or real situations. The example below illustrates the bridging function concrete materials can serve.

*Diane* (third grade): Diane was better than average at pencil and paper computations, but she was far below average in problem solving situations--especially those that required her to use mathematical ideas to describe real situations. She could use counting discs to "act out" a problem about "4 bags of apples with 6 apples in each bag," and she could also write an appropriate equation to find the number of counters in a 4 x 6 area; but she would get very confused when she was asked to translate directly from a real situation to written symbols.



Diane's LD teacher worked with her on the following kinds of activities:

(a) Practice breaking up difficult processes (like translating from real world situations into written symbols) into a series of simpler substeps--e.g., first translate from a real situation to a concrete model, then translate from a model into written symbols.

(b) For a series of real world problems, practice selecting an appropriate concrete model to act out the situation. Selections were made from among three or four alternative concrete models representing the three main types of elementary number situations: cardinal number situations, ordinal number situations, and measurement situations.

(c) Practice finding real world situations that are like three or four different kinds of prototype concrete models.

(d) Practice writing arithmetic equations to describe three or four of the most important types of prototype concrete models.

(e) Practice using three or four different types of concrete models to illustrate written arithmetic problems.

It was not enough for Diane simply to work with materials and then work with written symbols. She needed to practice translating from concrete situations to written symbols.

The best concrete instructional models are usually "half way" between written symbols and real situations. They are symbols in the sense that they can be used to represent many different kinds of real situations, but they are also manipulative objects that can be used to coordinate the systems of operations and relations that are related to a particular idea. Materials like Cuisenaire rods, arithmetic blocks, and geoboards are especially useful because the built-in structure of the materials makes it easy for children to "read-in" systems of mathematical relations that can later be coordinated and abstracted. Nonetheless, the fact that these materials are especially designed to represent many different kinds of real situations means that they can also cause difficulties for some LD children. Their symbolic characteristics often seem to stimulate children's imaginations in nonmathematical ways. For example:

> Mark (third grade): Mark was having great difficulty with arithmetic computation. His teacher believed that his lack of understanding of "regrouping" was the cause of many of his problems. Therefore, she began working with him individually using arithmetic blocks. The lessons were not very successful.
>
> In the first lesson, when she introduced the blocks, flats, longs, and units, Mark said that the block reminded him of a "death star" (from the movie "Star Wars") and that the flats were like Darth Vader's space ships. Mark disregarded the internal structure of the materials and focused instead on his own fantasies.
>
> Mark's teacher soon gave up trying to use arithmetic blocks to teach regrouping. She decided to try some materials and tasks that would put more restrictions on Mark's imagination. She chose to use a counting frame. But, just as in the case of the arithmetic blocks, the counting frame sparked Mark's imagination with a story about conveyor belts in a toy factory. Again the lesson was unsuccessful.

Dienes' book An Experimental Study of Mathematics Learning (1963) includes a number of examples involving children like Mark. Dienes writes:

> ...Interfering play may result from an overwhelming need to use the imagination actively. ...Such intrusions usually replace mathematical activity by some other activity where the imagination has freer play to express itself. It is perfectly true that to be a good mathematician it is necessary to have a good imagination and be well supplied with imagery, but this imagery must be disciplined in no uncertain manner before it will bear mathematical fruit. ...It is undisciplined imagination, giving rise to a flood of uncontrolled imagery, which is likely to get in the way of mathematical thinking. (p. 48)
>
> The problem appears to be to bring children face to face with structure; once they have tasted the excitement of coping with a structure and investigating how it works, playfulness will show itself rather as delight in coping than in overt play. (p. 55)
>
> The subject no longer 'plays' with the material, but 'plays' with an idea he has extracted from the material. (p. 107)

According to Piaget, mathematical operations and relations are abstracted from a child's interactions with his environment. But, the child's environment consists of more than sets of manipulative objects; it also includes sets of words or symbols. For example, some of the first number-related relational systems children use are manifested in their counting activities--where the objects being manipulated are number words. Similarly, for formal operational concepts, the new operations being learned are "operations on operations"--where the objects being manipulated are lower order operational systems. Therefore, for highly verbal and imaginative children like Mark (in the example above), it is sometimes helpful to use words or imaginary objects to help him organize various operational or relational systems. Multiplication ideas can be related to "skip counting," subtraction ideas can be related to "counting backwards" activities, and addition can be related to "counting on" activities. The objects the child manipulates do not necessarily need to be concrete objects.

Imposing structure on concrete materials involves translating from the world of mathematical ideas and structures to real world situations; and this translating consists of two sub-abilities: (1) identifying real situations to fit given relational or operational systems, (2) identifying appropriate relational or operational systems to describe real world situations. Sometimes these sub-abilities can be made easier by breaking difficult translation processes into series of simpler processes--that is, concrete models can serve a bridging function. But, this bridging function involves more than translating from mathematical systems to real situations; it also involves translating from one real situation (i.e., the model) to another. Unfortunately, this later ability can also be a source of difficulty for children.

Translating Among Various Models and Interpretations of an Idea. For any given mathematical idea, there are usually a variety of alternative mathematical interpretations and a variety of different kinds of concrete models corresponding to each interpretation. To select appropriate materials to teach a given idea, the following procedure can be used:

(1) identify a variety of mathematical interpretations for the given idea;
(2) identify several different types of concrete models corresponding to each interpretation;
(3) identify the system of operations or relations that is needed to understand each interpretation and each model;
(4) diagnose the operative ability of the student and select the interpretation and model that fits the student's ability level.*

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*Bell, Fuson, and Lesh's book Algebraic and Arithmetic Structures: A Concrete Approach for Elementary School Teachers (1976) gives a variety of interpretations and models for most ideas that occur in elementary school mathematics.

For example, in early elementary school, subtraction may involve three different interpretations (i.e., "take away," "comparison," or "missing addend") and three different models (i.e., measure models, ordinal number models, or cardinal number models). These various interpretations and models can be used to generate appropriate instructional materials for children.

To teach about subtraction, the following types of materials can be used: (a) a cardinal number model (e.g., counters), (b) an ordinal number model (e.g., a number line), (c) a measure model (e.g., Cuisenaire rods). Each of these materials emphasizes the "take away" interpretation of subtraction, and each is "good" in some ways and "not so good" in others. For example, Figure 4a emphasizes a slightly different interpretation of subtraction (Johnny had 5 balls and Sue took away 4. How many were left?); Figure 4b emphasizes the comparison interpretation of subtraction (Johnny was five feet tall. Sue was four feet tall. How much taller was Johnny than Sue?); and Figure 4c emphasizes the missing addend interpretation of subtraction (It was five blocks from Johnny's house to Sue's house. Johnny had already walked four blocks. How many more blocks must he walk?).

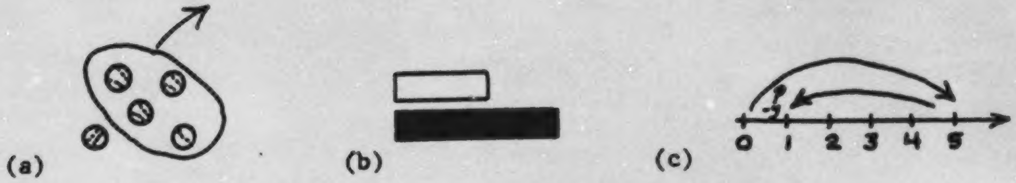
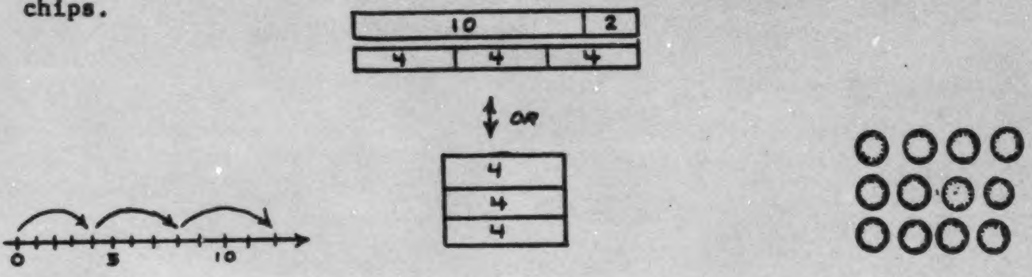


Figure 4. Illustrations of 5 - 4 = ☐.

Understanding a given idea not only means associating the idea with a given concrete situation, it also means recognizing the similarity among various concrete situations that embody the idea. Abstracting a mathematical structure from various embodiments essentially means recognizing an isomorphism between two structurally identical situations. That is, it means translating from one situation to another--looking at occurrences in the first situation to make predictions about occurrences in the second:

> Carol (fourth grade): Carol's remedial mathematics teacher had been using poker chips to illustrate basic multiplication facts. She had also used small "desk top" number lines and large "walk on" number lines on the floor. Carol seemed to understand how to act out simple multiplication problems using each of these models and she had also used each of these materials to act out addition situations. But when she was asked to translate from one model to another, she became confused. For example, when the teacher used poker chips to illustrate a given problem, and if Carol was asked to illustrate the same problem

using a number line (or Cuisenaire rods), she often gave incorrect answers. In fact, she even had difficulty translating from a large "walk on" number line to a small "desk top" number line, or from a situation involving real children to a situation involving poker chips.



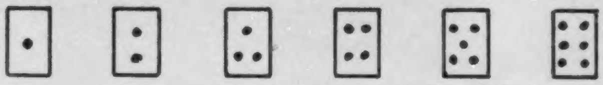
(a) a number line (b) Cuisenaire rods (c) poker chips

Figure 5. Illustrations of 3 x 4 = 12.

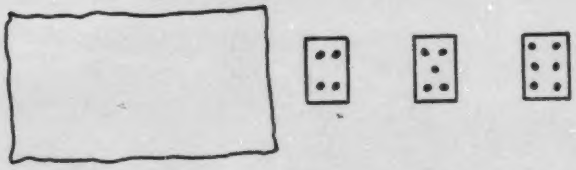
Adults usually recognize how the above illustrations are "alike," and can readily translate from one model to another--or from situations that are like one model to situations that are like another. But to many young children, these similarities are less apparent.

Translation difficulties often arise because different concrete models inherently emphasize different interpretations for a given idea. In these cases, difficulties arise not only because of lack of practice in translating skills but also because of lack of understanding about various interpretations and models for a given idea. For example, in cardinal number situations the physical properties of objects are often ignored (e.g., 5 elephants + 2 mice = 7 animals); whereas, in measurement situations, the object to be measured is often continuous--it is not partitioned into units but the units must be the same size. In cardinal number situations this is not the case. Ordinal number situations (first, second, third, etc.) emphasize still other number relationships. The example below illustrates differences between cardinal and ordinal number ideas.

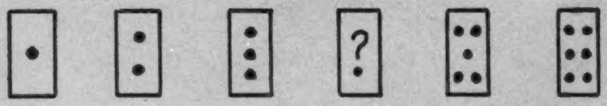
Tom (second grade): Tom's teacher put out a series of "dot pattern" cards like the ones shown below.



Then she covered the first few cards with a handkerchief and asked Tom how many cards were covered.



Tom had no idea how to figure out an answer from the cards that were showing. So, his teacher asked a simpler question. She removed the handkerchief and turned one of the cards face down. Then she asked Tom to guess how many dots were on the turned card.



Tom did not know the correct answer. He did not realize that the third card corresponded to the card with three dots, and in general he did not do well on tasks that required him to switch from cardinal to ordinal number properties.

Drawing explicit attention to the operations or relations that are involved in a task may not be the best way to help children learn to use these systems. It is one thing to organize reality using operation, relations, and transformation, and it is quite another to become formally aware of these operations. The intuitive mastery of a system of operations or relations is similar to the acquisition of an unconscious habit--what is at first a habitual pattern for using a system of operations to achieve some end later becomes a program in the sense that various substitutes can be inserted without disturbing the overall act. Forcing a child to become explicitly aware of the operations he is using may only be confusing. During the initial acquisition of mathematical concepts, children are not usually explicitly aware of the systems of operations they are using. This situation is similar to early stages in problem solving when children are able to solve problems but are unable to explain the steps that were taken to reach the solution. One is reminded of Mark Twain's yarn about the centipede who became instantly paralyzed when asked to explain how his legs moved.

Translating from one model to another does not require an explicit or formal awareness of the relations of operations that are embodied in the various models. Dienes (1963) writes:

> The rule-structure (in concrete situations) is not always consciously analysed, particularly not by young children; often it is stamped in and made acceptable and operational by repeated use and practice in recognizing situations where the rule-structure is applicable. (p. 157)

Translating from one model to another, or translating from mathematical systems to real situations, involves more than a few content-independent abilities. Specific information about various interpretations for a given idea and about various models for real situations is also required. Nonetheless, for some children, certain basic types of translation abilities seem to cause difficulty across a wide variety of specific mathematical ideas. Furthermore, instruction designed to improve these translation abilities frequently results in improvement across a wide range of mathematical content suggesting that these abilities may be fundamental processes that are needed in mathematical reasoning. They are processes that deal with hypothesized important links between figurative and operative aspects of thought, and they are processes that are seldom addressed by diagnostic tests or instruction. They are also processes that have caused difficulties for many LD and normal children.

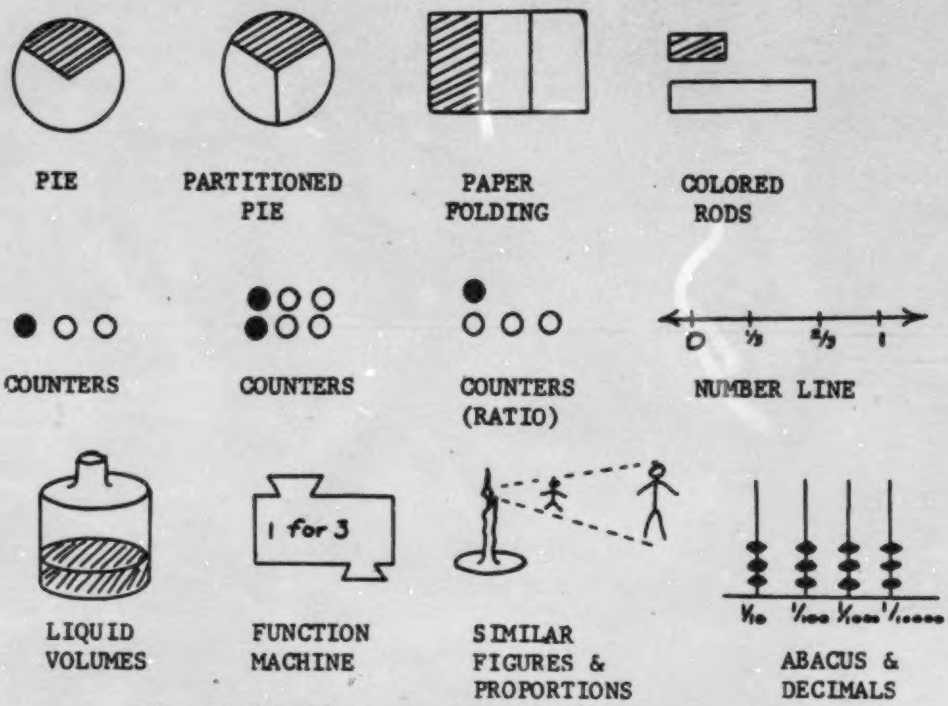
Abilities Associated with Spatial/Geometric Properties of Models. In addition to the translation processes that were discussed in the two preceding sections, there are other links between figurative and operative aspects of thought that may involve important mathematical abilities. According to Piagetian theory (Smock, 1973), the evolution of mathematical concepts typically involves both a figurative and an operative component (at least during early stages of development). It is well known that two tasks which are characterized by the same operational structure sometimes differ widely concerning the degree of difficulty (e.g., décalages). However, factors contributing to these variations have not been thoroughly investigated (Laurendea & Pinard, 1970). What is known is that situations are facilitating (or confusing) to the extent that there is some (or no) immediate connection between the figurative structure of the task and the operative structure of the concept involved.

Logical, arithmetic, and geometric concepts each arise out of a common source, which is children's interactions with their environment. Because spatial experiences tend to dominate children's interactions with concrete materials, it would seem sensible to investigate the extent to which geometric experiences could facilitate or hinder the acquisition of arithmetic concepts. Unfortunately, most research investigating relationships among spatial abilities and arithmetic abilities has not focused on the kinds of spatial abilities suggested by Piagetian theory.

Piaget (1965), Dewey (McLellan & Dewey, 1914), and several Soviet psychologists (e.g., Gal'perin & Georgiev, 1969) have described ways that misunderstandings concerning number ideas are closely linked to a lack of understanding of certain geometric notions. For example, Piaget's number conservation task tests whether children realize that the number of objects in a set is invariant under simple spatial displacements (i.e., geometric transformations). Tasks such as these show that logical, arithmetic, and geometric notions are not initially learned as distinct categories of concepts. Rather, for young children, these three types of ideas exist in a confused and overlapping state and only gradually become differentiated and coordinated. Young children tend to confuse judgments about: (a) the number of objects in an array of circles, (b) the density of the configuration, (c) the area covered by the array, and (d) the length of the rows or columns. Similarly, objects that are logically alike are often confused with objects that are spatially close together.

Most of the models (e.g., number lines, arrays of counters, fraction bars, Cuisenaire rods, etc.) and diagrams teachers use to illustrate arithmetic and number concepts presuppose an understanding of certain spatial/geometric concepts. Consequently, because of a lack of understanding (or misunderstandings) about spatial geometric concepts, children often develop misunderstandings about models that are used--and about the number ideas the models are supposed to illustrate. For example, in the upper grades, number lines and "area" or "volume" models are used to introduce fractions, and similar triangles are used to illustrate proportions. Yet, there is abundant evidence (e.g., Gal'perin & Georgiev, 1969; Piaget & Inhelder, 1967) that children frequently have problems understanding each of these models. Nonetheless, very little work has been done to isolate the geometric concepts that the models presuppose or to identify links between misunderstanding of models and misunderstanding of the ideas they are intended to illustrate (e.g., Lesh, 1976).

In Figure 6, each of the materials can be used to illustrate rational number concepts, each emphasizes different aspects of the number 1/3. Some of the materials stress the "fraction" or "part of a whole" interpretation of rational numbers. Others emphasize the "ratio" or "proportion" interpretation of rationals. And, others emphasize the "ordinal" or "operator" interpretation of rationals. Still others illustrate rationals as "ordered pairs" or as extensions of our numeration system.



PIE
PARTITIONED PIE
PAPER FOLDING
COLORED RODS
COUNTERS
COUNTERS
COUNTERS (RATIO)
NUMBER LINE
LIQUID VOLUMES
FUNCTION MACHINE
SIMILAR FIGURES & PROPORTIONS
ABACUS & DECIMALS

Figure 6. Illustrations of 1/3.

Other issues also arise. For instance, among the illustrations in Figure 6, which materials are most abstract, or concrete, or complex? Which will be easiest for youngsters to use? Which materials will allow youngsters to deal most directly with the most elementary interpretations of rationals and yet not lead them to form misconceptions that will make higher order understanding more difficult (e.g., youngsters who have learned that rational numbers refer to "parts of a whole" may find difficulties when they confront three-halves). What role does familiarity play in selecting materials? Which materials will draw upon more useful intuitive notions without also conjuring up irrelevant properties? How many different types of materials should be used, and in what order should they be presented? Finally, are there any generalizations that can be made about discrete models versus continuous models, or about cardinal-versus-ordinal-versus-measurement models? These questions make it clear that even within the realm of geometric figures and "real world" materials, concrete-to-abstract and intuitive-to-formalized dimensions must be considered. Some of these dimensions of intellectual growth may be related to the development of important mathematical abilities.

Little has been done to investigate how the figurative content of a problem affects the difficulty of mathematical tasks (Lesh, 1978). Piaget has focused on the operational aspects of tasks and concepts, but he has deemphasized the figurative aspects. The kinds of spatial abilities LD specialists have investigated are not the type that are likely to clarify relationships between figurative and operational aspects of thought. The influence of figurative content on operational ability is important information for teachers who must devise models to illustrate mathematical concepts and it seems likely that some important mathematical processes may be involved. More research is needed concerning relationships between figurative and operative aspects of thought.

## The Role of Activity in Mathematics Education

The most obvious justifications for activity in mathematics instruction derive from Piaget's processes of assimilation and accomodation. The learner is viewed as an active agent who interprets the environment using internal models or structures which are gradually modified to "fit" progressively more complex situations. In mathematics the structures people use to interpret reality consist of organized systems of operations, relations, or transformations; and the construction of these operational systems requires other types of activity in addition to those associated with assimilation and accomodation.

> "Operations (i.e., operations, relations, or transformations) are internalized schemes of actions, that are reversible and that exist as part of a system that is characterized by laws of totality." (Beth & Piaget, 1966, p. 234)

The above definition implies that: (i) operations, relations, and transformations are abstracted, not from concrete materials, but from interactions between a student and his environment--e.g., interactions between a student and concrete materials, or interactions among students, and (ii) operations, relations, and transformations do not exist in isolation. That is, they exist only as part of a coordinated system (e.g., a grouping) that also involves other operations, relations, or transformations.

Both of the above statements involve activity, first because operations are abstracted from actions (although there may not be motor activities), and second because the actions take on the status of operations only when they are modified by being treated as part of a whole system of actions. Furthermore, these two types of activities are linked to abilities that may be deficient in some LD children.

The Ability to Coordinate Systems of Operations or Relations. Abstracting operations from one's own actions consists not simply of taking note of isolated interactions; it requires the reconstruction of these actions on a higher plane. Individual interactions gradually take on new significance (Piaget calls this reflexive abstraction) as they are modified by being treated as part of a whole operational structure. The evolution of operational structures does not begin with individual isolated operations which are successively linked together. Rather, the evolution of structures of operations occurs simultaneously with the evolution of the operations that the structure subsumes. Both the structure and its operations simultaneously crystallize out of a system of schemes of actions as it becomes progressively coordinated (genetic circularity).

While the coordination of a system of schemes of actions is achieved progressively, its completion is marked by a momentary acceleration in this construction as the child shifts to a qualitatively higher level of thought. As a result of this reorganization, new self-evidence typically appears with regard to concepts whose definitions depend upon the application of the given structure. In this way, certain operational concepts (such as the concept of a series, or class) and certain properties (such as transitivity) arise out of structured wholes of operations, the completion of which explains the necessity of its elements insofar as their meanings are dependent on that whole.

To get children to master a system of operations, one of the main problems is to get them to coordinate a system of actions into an operational whole. However, it is precisely this coordination that preoperational children and some LD children lack.

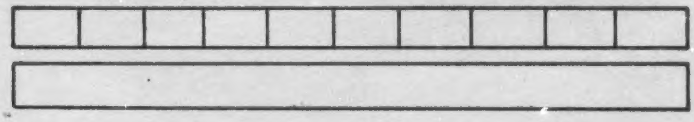
> Karen (sixth grade): Throughout elementary school, Karen had been a good student in every subject except arithmetic. Her mathematics skills were evaluated to be at the second grade level. She had also developed a strong fear and dislike for nearly everything called arithmetic or mathematics so she would prompty "tune out" whenever her teachers or parents tried to work with her. Her school's LD specialist decided to start working with Karen using games in which mathematics skills would be used and to work on topics that Karen might not recognize immediately

as being mathematics--e.g., topics like measurement or geometry where basic quantitative judgments could be practiced.

In board games, Karen seemed to have no intuitive "feeling" for the relative size of seven, seventeen, seventy, or seven hundred. In monopoly-type games, Karen was unable to estimate how far a given roll of dice would take her, and she showed little awareness of the significance of different dollar values for properties.

Because Karen had great difficulty measuring with a ruler, her teacher tried to teach her to measure things (e.g., desk tops, books, etc.) by laying orange Cuisenaire rods end to end, But, when the teacher showed Karen that her desk was 8 rods long, and then asked Karen to find the width of her desk, Karen was unable to put the rods in a straight line from one side of the desk to the other. When she succeeded in making a straight row, the endpoints did not fit the sides of the desk; when she made the endpoints fit the sides, the rods either had large spaces between them or else they overlapped or were out of line. In general, whenever Karen paid attention to the endpoints of the whole row, she neglected to notice the arrangements of individual rods. When she paid attention to individual rods, she neglected to notice the whole configuration. Similarly, in other measurement or number tasks, she had difficulty keeping parts and whole configurations in mind at the same time. For this reason she failed most "part-whole" Piagetian conservation tasks involving length, area, or number ideas.

Karen's teacher cut a piece of adding machine paper to make two strips of paper that were exactly the same length (one meter). Then she put the two strips on a table approximately 5 centimeters apart and drew decimeter lines on one of the strips. The teacher was hoping to glue orange rods to the two strips to help Karen measure things. But, when the lines were drawn on one of the strips, Karen no longer believed the two strips were the same length. The inability to coordinate parts and wholes was one example of Karen's general difficulty in coordinating systems of relations or operations.



How do we get children to coordinate a given system of operations? A possible answer is, "pretty much the same way we get them to coordinate (and then think back about) any system of actions;" we start by putting the child in a situation where the system of actions will be easy to use and we gradually introduce more and more complex situations and more elaborate systems of actions. The process is similar to helping a youngster coordinate the act of hitting tennis balls--except for one fact. To get children to coordinate a system of mathematical operations, the situations we pose must usually involve activities where there is some similarity (i.e., isomorphism) between the structure of the task and the operational structure the child is supposed to coordinate. For example, "putting a set of Cuisenaire rods in order" can be used to teach children about the relation "bigger than" whereas gesturing games (e.g., stretching up on tippy toes) are unlikely to contribute to the coordination of the relevant system of relations.

Not all activity is helpful in learning mathematical ideas. For instance, hyperkinetic children are no more desirable in a mathematics laboratory than they are on a tennis court. The goal is to get children to coordinate a system of actions, and then to reflectively abstract the individual acts within the system. Activity purely for the sake of activity is not necessarily conducive to cognitive growth. For example, recent research studies at Northwestern (e.g., Musick, 1978; Schultz, 1978) have shown that, for two tasks which differ only in the degree of activity that is required of the child, "high activity" tasks are often consistently and significantly more difficult than the low activity tasks. These studies also give examples to demonstrate a variety of problems that can arise from activities in which the structure of the tasks is unrelated to the structure of the concept being taught. The kind of activities that seem most conducive to learning a given idea are those which are isomorphic to the system of operations that characterize the idea.

The Ability to Reverse Thought Processes. In Piagetian theory, the key to the emergence of a whole system of operations is the appearance of the inverse of the operation. This reversibility phenomenon (i.e., the ability to reverse a given relation or to undo the effects of a given operation) is critical in the development of many operational structures. Reversibility is attained when a child ceases to think in terms of isolated operations--or results of isolated operations, and begins to think in terms of systems of operations--and of invariant properties under systems of operations.

Reversibility appears to be related to the operational thinking of students at all levels. For example, among the abilities of gifted mathematics students, Krutetskii (1976) lists the following:

(1) Ability to switch rapidly from a direct to a reverse train of thought. "The capable pupil's thought wanders freely, if needed, from a straightforward course; it is easy for an able pupil to pass from a direct to a reverse train of thought, which presents difficulty for average pupils. From an ordinary proof to proof by contradiction--from a direct to a converse theorem--these transitions are made by capable pupils without difficulty." They develop the ability to switch from direct to reverse operations. (p. 187)

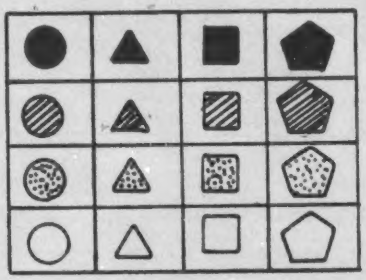
(2) Flexibility of thinking. "Capable pupils can quickly transfer

> from one aspect of a discussion to another, from one method of approach to another, from one method of solution to another. A surprising mobility of thought distinguishes my capable pupil G. Kh. He tries this way and that." (p. 188)

> (3) Curtailment of the reasoning process. "In capable pupils the reasoning process is curtailed and is never developed to its full logical structure. This is very economical, and in this lies its value...I have often observed how a capable pupil thinks: for the teacher and the class it is a detailed process, with all the links in the sequence, and for himself it is fragmentary, cursory, very abbreviated, a shorthand record of thought." (p. 189)

Through his work involving younger children of average ability, Piaget has shown that flexibility of thought--including the ability to foresee difficulties and take "short cuts" in reasoning processes--are closely related to the attainment of reversibility of basic operational or relational structures. Lack of reversibility and rigidity of thought also are quite common among LD children who have not coordinated the operational systems required for particular mathematical tasks. The example below illustrates one such child.

Johnny (first grade): Johnny was trying to organize geometric shapes into a 4 x 4 "checkerboard" matrix. The game involved four shapes and four colors as shown below.



Each time Johnny tried to complete the task he would get one or two rows correct, then he would make a mistake. But, when he made a mistake, he could not "back-track" to correct it and then continue; he would "mess up" the board and start all over from the beginning.

On other single problem-solving tasks, Johnny's thinking resembled a movie that could only be run from start to finish in the forward direction. If a difficulty arose somewhere in the middle of a solution attempt, Johnny would not backtrack to a point where he could proceed; he would start all over from the beginning. Unlike other children whose thinking more closely resembled a series of computer-like subroutines that could be coupled together in flexible ways, Johnny either completed a whole problem solving process without an error, or else he did not solve the problem at all.

Kim (third grade): Kim had been having difficulty in arithmetic throughout grades 1 and 2. But, she had always been good in other subject areas and seemed able to get along in arithmetic. However, in third grade, more serious difficulties became apparent. Kim's teacher noticed that she had extraordinary difficulty on subtraction problems which required the use of different "borrowing" subprocedures. Later, the LD specialist in Kim's school found that Kim had difficulty following directions in many situations that involved linking together several subprocedures to accomplish some overall task. When the subprocedures involved choices among several alternatives, Kim would often lose track of the overall goal.

Irreversibility and inflexibility have probably played significant roles in several recent mathematics learning studies involving LD subjects. For example, a study by James (1975) involved 40 seven-year-old LD children and 40 seven-year-old normal children. In each group, every child was given a series of measurement-related Piagetian tasks. The major purpose of the study was to investigate the relative difficulty among tasks for children in each of the two groups. For each child, error patterns and solution procedures were also recorded. One interesting side result of the study showed that among LD children who solved given measurement tasks correctly, the solution time was often significantly shorter than the solution time among normal children who did the task correctly. For example, on one measurement task, 24 normal children solved the problem whereas only 6 LD children solved it correctly. Yet, among the 6 LD children, the average solution time was 24 seconds with no one requiring more than 1 minute and 30 seconds, while among the 24 normal children, the average solution time was 2 minutes and 12 seconds with no one solving the problem in less that 1 minute and 30 seconds. Data similar to these were obtained on other tasks in James' study. Similar results were also observed by Lesh in a series of pilot studies involving constructive measurement tasks, computation tasks, and geometric construction tasks. The numbers of LD subjects in these pilot studies were not sufficient to justify any sweeping conclusions; nor were the studies sufficiently controlled to isolate causal connections for particular behaviors. But, follow-up questioning of individual LD children did consistently show that even the successful solution procedures often tended to be rigid, inflexible, and irreversible processes rather than the kind of flexible sequences of subroutines that characterize the successful solution procedures used by most normal children.

In the above situations, speed in giving answers was not necessarily good. Impulsivity, rigidity, inflexibility, and irreversibility all contributed to quick solutions for particular problems, but these characteristics seldom contribute to the long range developmental problem solving abilities.

Krutetskii (1976) writes:

> It is not a matter of 'speed' when we are discussing ability, 'rate of progress,' or 'tempo of progress.' A person can work slowly but can progress quickly in learning a task, and vice versa. Time and again we shall remark on the necessity of distinguishing between one's individual tempo of work and one's tempo of progress, between the rate of work and the rate of development. (p. 62)

The Ability to Form Relationships Among Various Operational Systems. The ability to coordinate operational systems for individual tasks is closely related to the ability to recognize structural similarities among several "related" tasks. For example, Lesh (1975) developed three sequences of tasks (denoted S1, S2, S3, ..., S6; N1, N2, ..., N6; and C1, C2, ..., C6) which were graded in difficulty and which pertained to Piagetian seriation, number, classification ideas respectively. The tasks were selected in such a way that, for each of the three sequences, the probability would be small (i.e., 15%) that a child would correctly respond to a higher order task (e.g., S(n+1)) before he is able to respond to task S(n). That is, the three sequences were derived so that most children would master the tasks in a relatively invariant order (e.g., task S1 before task S2, task S2 before task S3, etc., and similarly for the other two sequences). The goal of the study was to compare relationships among children's progress through the three sequences. The results of the study, which involved 160 normal kindergarten children, showed that these children progressed through the three sequences in a parallel fashion. That is, a child who was at level three on the seriation sequence was usually at approximately level three on the number and classification sequences. The chances were small (i.e., 15%) that he would be more than one step higher or lower on either of the other two sequences. Later the study was replicated with 40 LD children of approximately kindergarten age. The results of this follow-up study showed that, unlike the normal children, more than half (i.e., 23) of the LD children were "out of phase" on the three sequences. For example, a child might rank quite high on the number sequence, but quite low on the seriation and classification sequences.

James (1975) used a design identical to the one described above except that subdivision, measurement, and change of position sequences were used instead of seriation, number, and classification. The results in James' study were similar to those obtained by Lesh. That is, operational systems that develop syncronously in normal children do not necessarily develop together in LD children.

The implications of the above results for Piagetian theory are not entirely clear. Piaget has argued that number ideas result from a synthesis of seriation and classification concepts and that measurement ideas result from a synthesis of subdivision and change of position concepts. And the empirical facts supporting Piaget's argument have been replicated with success in a variety of cross-cultural studies. However, LD subjects seem to represent a population that does not behave in the usual way on batteries of Piagetian tasks. What can we conclude about a given LD child's concept of numbers if he can perform the usual conservation tasks correctly but is unable to give correct responses to related seriation and classification tasks? Perhaps the operational structures of LD children's concepts are not exactly the same as those for normal children; or perhaps the structures are simply not as closely related to one another. Perhaps, because of some underlying disabilities, the general mathematical experiences of some LD children are sufficiently different from those of normal children that entirely different operational systems are developed to make judgments about number, measurement, arithmetic, and geometric concepts.

Piagetian interviews have quite frequently shown that many LD children have failed to coordinate some of the important operational systems that their peers can use to make mathematical judgments. Furthermore, the systems that LD children have coordinated often seem to be organized in a way that is unusual compared with normal children. Perhaps these operational systems represent alternative conceptualizations that are just as valid as the ones used by most normal children; or perhaps they represent misunderstandings that must be corrected through the organization of new operational systems. More research will be needed to resolve these issues. This is a research area in which information about the abilities of LD children could significantly alter the way we think about the cognitive abilities of normal children.

## The Role of Problem Solving in Mathematics Instruction

Piaget does not explicitly give his blessing to problem solving--at least not if the problems are the type that are usually encountered in school books. He does claim that cognitive growth occurs through an equilibration process involving assimilation and accommodation and that disequilibrium situations will provide a propellent to cognitive growth. The problem, like a carrot for a donkey, must be just close enough to be partly assimilated and just far enough to require some accommodation. However, many disequilibrium causing situations are not "problems" in the normal sense of the word; and many school "problems" either do not require an accommodation or else cannot be assimilated by many students. So, for Piaget, problem solving contributes to learning insofar as it facilitates the equilibration process.

Disequilibrium occurs when two competing interpretations of an event are in conflict; and conflicting interpretations occur through two competing assimilation processes--generalizing assimilation and discriminating assimilation (Piaget, 1971). Saari (1976, Note 4) has formulated a mathematical model describing how these two types of assimilation fit together with the accommodation process to produce cognitive growth and has also described how these processes are related to problem solving. Readers who would like a detailed interpretation of Piagetian "problem solving" are referred to Saari's papers. The following general ideas will be sufficient for the purposes of this paper.

For Piaget, cognitive structures evolve out of lower-order structures, and are gradually subsumed into higher structures. That is, at any given stage of development, a given structure is a form for lower order systems and content for higher order systems. If a child focuses on lower order subsystems to interpret an event, then he is using discriminating assimilation, and if he focuses on higher order subsuming systems to interpret the event, then he is using generalizing assimilation. If the total structure of the problem is well coordinated (i.e., if it is equilibrium) then the two interpretations will fit together and will not conflict. If the two interpretations do conflict then an accommodation will be needed to reconcile them. That is, new relationships must be constructed to coordinate the two interpretations.

What mathematical abilities are implicit in the equilibration/assimilation/accommodation process? The following three examples illustrate disabilities that are associated with dysfunctions in generalizing assimilation or in discriminating assimilations.

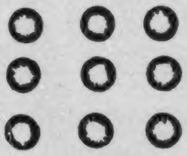
(1) Generalizing Assimilations.

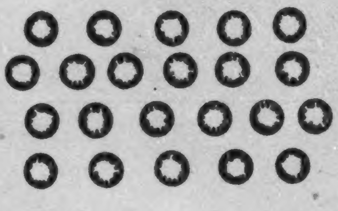
Mark (third grade): Mark was already mentioned in Part II of this paper. He was the youngster who thought arithmetic blocks looked like space ships, robots, and death stars from the movie "Star Wars," and whose teacher was unable to get him to investigate the internal structure of the materials. In these situations, Mark was assimilating parts of the instructional situations that his teacher presented. That is, given almost any set of materials, he would quickly make up a fanciful story telling how these materials were like other situations that he understood and found interesting. But his generalizing assimilations usually disregarded the internal structures of the instructional materials.

Ron (fifth grade): Ron's LD teacher was trying to help him learn about measuring lengths and areas. She was using a Socratic method of questioning to get Ron to think about the details of various measurement problems. But, Ron was very skillful at generating sequences of nonsequitors that prevented his teacher from examining any given situation in detail. Ron would flit from one situation to another, or from one issue to another, without thoroughly learning any of them. After the session, Ron's frustrated teacher explained, "Trying to talk to Ron about measurement is like trying to talk to my grandmother about religion or politics. He hops around a set of loosely related topics but refuses to think about any of them in detail."

(2) Discriminating Assimilations.

Marci (second grade): Marci was doing quite well in every subject except arithmetic. Her LD teacher was working with Marci on number activities involving one-to-one matching among sets and on other classifying and ordering tasks. On one activity Marci had been given a box full of 20 poker chips. She was asked to copy a 3 x 3 array of nine poker chips. Marci began by making a row of chips. Then she made a second and third row. But each row had far too many chips, and she became so involved in the act of putting out chips that she forgot her original goal and simply put out all of the poker chips in her box.





When Marci had put out all of her poker chips she was asked if her configuration had the same numer as in the 3 x 3 model. Marci knew that her configuration had more. So, she pushed all of her chips back into her box and began the task from the start. Nonetheless, on her second attempt she made the very same mistakes as in her first attempt. When she got into the details of a problem, she would lose sight of the overall goal.

Generalizing assimilation, discriminating assimilation, and accommodation are each dependent on a child's ability to coordinate given operational structures. That is, a child does not become a "good generalizing assimilator" in all situations. The ability to assimilate is always structure specific, and the fundamental goal of instruction is to help the child to develop the relevant structures. Nonetheless, some children seem to have an almost permanent imbalance in favor of one form of assimilation, neglecting the other form of assimilation. That is, given a new problem (situation, idea, or example), some children have a strong tendency to neglect or distort the details of the problem in order to fit their own subsuming classification schemes. On the other hand, other children continually get embroiled in the details of new problems or ideas and consistently fail to "see the forest because of the trees." Perhaps these assimilation biases are the result of some underlying disability, perhaps they are the result of some learned behavior, or perhaps they are closely related to some social/affective factors or self concepts. More research will be needed to clarify these issues. Minimally, research in this area should include an accurate assessment of the operational structures available to given children and an accurate assessment of the operational structure of the new idea.

What is it, beyond having a concept, that allows a normally intelligent person to use the idea to deal with math-related problems in everyday situations? Piagetian studies have shown that the operational structure of an idea is an important factor determining the difficulty of the idea. Geometry research (e.g., Fuson & Murray, 1978; Schultz, 1978; Thomas, 1978) has shown that the figurative content of the task situation can also radically influence the difficulty of a given problem situation. And, a student's proficiency with certain problem solving processes also influences the difficulty of tasks. However, the relevant processes do not necessarily correspond to the typical kinds of problems solving processes discussed by Polya (1945), Krutetskii (1976), and others.

As was mentioned earlier, most information about problem-solving processes has come from situations involving older students, exceptionally bright students, individual students working in isolation (often in artificial laboratory situations), or situations involving highly contrived word problems, mathematical puzzles, or proofs. Elementary school children, average (or below average) ability students, and applied problem solving processes have been neglected. For this reason, the "problem solving" processes educators discuss often seem inaccessible to younger children or less gifted students, and applied problem solving processes like modeling have been ignored. Some of these applied problem solving processes will be discussed in Part IV of this paper.

## The Role of Small Group Activities

When educators talk about problem solving situations, they often ignore the fact that most people work on real-world problems when other people and other resources are available. People seldom work in isolation using only the power of their own minds to solve problems. Instead, good problem solvers learn to amplify their own powers through effective use of outside resources. For example, when real people solve real problems, one of the most often used problem solving strategies is to "ask someone who can give the needed information." This is not to say that good problem solvers solve most problems by asking someone else to do their work for them. Formulating a problem in such a way that a specific bit of information can be requested is not a trivial skill. In fact, one of the most obvious characteristics of good problem solvers is that they are good question askers. Once a question is formulated in a nice way, answer giving is often quite easy.

Many individual problem solving strategies are quite difficult for average or below average ability youngsters. But when these internal processes are externalized in the context of small group activities, they are often easier to describe in a form that is understandable to lower ability problem solvers. For example, problem solving strategies like "consider a similar problem," "consider an auxiliary problem," or "consider a special case," can be summarized with the simple advice, "look for a related problem." Yet, to poor problem solvers, this advice often seems quite foolish because, "I already have one problem I cannot do, I do not need another." To poor problem solvers, a more sensible suggestion is, "Look at the same problem from a different point of view."

In several recent research studies (e.g., Cardone, 1977) where groups of four students were supposed to work together on problems, individuals often worked independently--each conceptualizing the problem in quite different ways, and each unconscious of misleading biases inherent in his own point of view. This is one reason why "brainstorming" is often a useful problem-solving technique.

In group "brainstorming" sessions, students can be bombarded with a variety of different ideas and approaches, and can simultaneously become more self-critical about their own points of view. They can also be made to notice: (a) some people are good talkers while others are good listeners, (b) some people are good generalizers while others are better at working out details, and in general (c) a variety of different roles are beneficial to good problem solving. Good problem solvers must be flexible enough to switch quickly from one role to another while solving a problem.

Many other problem solving strategies are greatly simplified in group situations. Problem solving strategies like "identify the givens," "identify the unknowns," and "eliminate irrelevant information" all having to do with the general recommendation, "understand the problem." However, this advice again seems rather useless to poor problem solvers whose superficial understanding of the problem often leads to selecting or eliminating information on rather artificial bases. On the other hand, except for specific recommendations about identifying knowns and unknowns, it is difficult for poor problem solvers to understand what it means to "understand the problem." It is much easier to say, "Use your own words to describe the problem to a friend," or "Describe some other problems like this problem." Poor problem solvers sometimes flounder with a problem for a long time before noticing (if asked) that they are unable to give a clear description of the problem to a friend. So, once again, group activities can force students to "understand the problem" and "organize the information given." Eventually, they may become self-critical enough to work on problems and no longer need group work to overcome their subjectivity and egocentrism.

When students work in groups to solve problems, they often see that many problems can be solved in a variety of ways--some of which are "better" than others. In fact, in research with gifted youngsters (e.g., Krutetskii, 1976) the hallmark of outstanding problem solvers is not so much whether answers are right or wrong but whether "clever" procedures were used. "Good" problem solvers are flexible thinkers who are capable of solving problems in several different ways; so, when one path is blocked, another route can be taken.

The above points are not intended to imply that we should explicitly _teach_ group problem solving techniques. Rather, group problem solving situations furnish an effective context to teach individual problem solving procedures--especially when the problem solver is at a relatively primitive level of skill acquisition.

If a person has not yet coordinated a given set of operations, he will generally have the following cognitive characteristics when he is forced to make judgments which rely on the use of this system: (1) centering--i.e., he will not "read out" all of the information that is available; he will focus on only the most obvious features of the situation and will fail to notice less obvious features; (2) egocentrism--i.e., he will "read in" meaning and information because of his own preconceived biases; he will distort the situation to fit his own understanding even when his own ideas do not correspond to objective reality. To overcome both of these tendencies (i.e., centering and egocentrism) children may find it helpful to work in groups where they are forced to coordinate their own point of view with that of other children. While one child may center on one aspect of a situation, another child may center on another; and children with various idiosyncratic interpretations of a situation will be forced to confront one another. On the other hand, it is well known that preoperational children--precisely because of their egocentrism--often function in groups without any real interaction taking place. Two-year-old play groups are often characterized by parallel play in which each child carries on a monologue in the presence of the other children. Preoperational children tend to be quite unimpressed with apparent (to an adult) conflicts between their own interpretations and those of other children--or between their own interpretations at one moment compared with another.

Even at preoperational levels of development, group activities can be beneficial. According to Piaget, cognitive development is characterized not only by a concrete-to-abstract dimension but also by an external-to-internal dimension. That is, actions on real objects are gradually internalized to form coordinated cognitive systems. So a teacher can concretize a given operation and he can also externalize an operation. For example, if a preschooler is learning to put Cuisenaire rods in order, the child may get so involved in the details of the task that she has difficulty keeping in mind the overall task. In such a case, a child who finds it difficult to build a Cuisenaire rod "staircase" may still learn a great deal from watching the teacher (or another child) build a staircase. Watching can also involve activity--just like tasks which demand more overt action.

Several Soviet psychologists (e.g., Gal'perin, Vygotskii, and Leontiev) have written extensively about the internalization process and its relationship to the development of operational or relational thinking. These studies furnish many examples showing that even at sophisticated levels of development centering and egocentrism phenomena are still present in the internalization process. For example, persons reading a new mathematics textbook for the first time will center on some points and neglect others; and they will reinterpret and perhaps distort many ideas in order to fit their previous conceptualizations of the subject. Similarly, in problem-solving settings, a good problem solver learns to behave as though he were, within himself, several people sitting around a table working together to solve a problem. He is objective in the sense that he sees the problem from multiple perspectives and he is aware of (and self critical about) his own perspective at any given moment. In this way he is forced to be more analytic and to attend to more aspects that are implicit in any single point of view.

In addition to previously mentioned cognitive justifications for small group activities, a number of special affective justifications are also apparent. In part learning to be a problem solver means acquiring a problem-solving personality. One of the first characteristics of a good problem solver is that he interprets an unusual number of daily situations as problems--that is, as situations where his problem solving skills may be relevant. One of the first steps a good problem solver takes is to identify a given problem as "do-able" or "un-do-able," and next as "easy" or "difficult." Then appropriate solution strategies are selected to fit the initial appraisal. However, it is quite obvious that people who are good problem solvers in one context, in one type of situation, or in one discipline may be average or below average problem solvers in another.

In modern psychology it has become more and more clear that cognitive adaptation exists in an ecological system with other adaptation-seeking mechanisms and is influenced by them. Learning, socializing, and adjusting cannot be completely separated. Much that we now call learning is social learning. Many aspects of human learning that we have traditionally regarded as "cognitive" development are specializations of "social" development. The central issues are not so much about how the child develops knowledge, but rather, about how he develops shared or cooperative knowledge that creates not only the objective (i.e., intersubjective) reality, but also the individual's conception of the self (Mead, 1934) and his moral ideology (Kohlberg, 1969).

Research has shown that children may well form severe personae for different situations--family, peer groups, school. The child's "personality" is different at different times, in different situations, in different mental and physiological states, as a function of cognitive, social, and emotional load, and as a function of his particular agenda at the moment. Cognitive performance is moved upward and downward by load factors in the child and in the situations--e.g., noise, emotionality, distraction, confusion, shyness, anxiety. So children fluctuate in their apparent ability depending upon time and place. Psychological descriptions of people have for a long time centered around the premise that people have traits, personality traits and cognitive traits, that endure and that are manifest in all situations. This notion is currently under strong attack, particularly in personality theory.

## Summary

What abilities have been suggested in the preceding section? Some possible candidates include: (a) the ability to decenter, (b) the ability to overcome egocentrism, and (c) the ability to internalize. But, each of these potential "abilities" results from a more basic type of ability--the ability to coordinate a given system of operations and relations. For example, decentering is not really an ability at all because a person never outgrows her tendency to center in some situations. Even the most highly educated genius will center in situations that require her to use a system of relations she has not yet coordinated. Piaget himself centers in situations that require structural systems that he has not yet organized.

Reversibility, centering, egocentrism, and equilibrium are all ideas that refer to particular systems of operations. So, the fundamental problem is to help children coordinate the systems of operations that are needed to make judgments about particular ideas. Then, the child will decenter, will be less egocentric, and will exhibit reversibility in her thinking about these ideas.

In Part II of this paper, each of the three categories of abilities discussed in connection with Gagné's theory was related to a child's organization of a system of mathematical ideas. Gagné's theory focuses on between-idea systems that describe how individual ideas are related to one another; in particular, Gagné focuses on the influence of prerequisite ideas. Piaget's theory focuses on a second type of system--within-idea systems. That is, to make judgments related to most mathematical ideas, children must learn to use organized systems of relations or operations. Consequently, among the abilities that were mentioned in Part III, most were related to a child's ability to organize given operational systems.

One of the characteristic features of mathematics is its structure, including systems of both the within-idea and between-idea variety. Therefore, it should not be surprising if the kind of abilities that are particularly important in mathematics have to do with a child's ability to organize and use these relational and operational systems. Part IV of this paper will focus on a third type of relational system--relations among various representational modalities. This latter type of relational system has to do with another salient characteristic of mathematics--its distinctive use of symbols.

## IV. Abilities Associated with Bruner's Theory of Learning

During the past ten years, Bruner's theoretical interests have seemed less relevant to mathematics instruction than his work in the late 1950's and 1960's. Nonetheless, his ideas have had important influences on the major curriculum reform movements that shaped today's mathematics curriculum and some of his recent work in psycholinguistics (1975) has begun to return to issues that attracted the attention of mathematicians and natural scientists a decade ago. In the 1960's, Bruner's views about learning and instruction were appealing to mathematicians because of the central role he assigned to the structure of scientific disciplines, because of his treatment of scientific intuition, and because of his emphasis on process as well as product objects for instruction. These three factors are also important to a discussion of mathematical abilities. The kinds of processes Bruner discusses go beyond the kind of problem solving processes that mathematics educators usually stress. The instructional relationships Bruner emphasized differ from those discussed by other "learning theorists" like Gagné and Piaget. In addition, Bruner's emphasis on intuitive learning is quite different from the emphases of most instructional theories that are popular in mathematics education or in special education.

Bruner also stresses several aspects of mathematics learning that have not been adequately addressed by current "high activity" psychological theories. Neglected areas include the influence of figurative content on operative ability and the influence of spoken language on written-symbolic mathematical understanding. Furthermore, he has organized these neglected areas into a conceptual framework that is useful in order to discuss mathematical abilities that may be deficient in LD children.

Two of the most obvious distinguishing characteristics of mathematics are its structure and its distinctive use of language and symbolism. Therefore, if abilities that are uniquely important in mathematics exist, they would likely involve one of these two characteristics. Bruner's writing emphasizes both of these characteristics and describes their possible influence on mathematics learning.

Bruner claims:

> Grasping the structure of a subject is understanding it in a way that permits many other things to be related to it meaningfully. To learn structure, in short, is to learn how things are related. (Bruner, 1960, p. 7)
>
> The merit of a structure depends upon its power for simplifying information, for generating new propositions, and for increasing the manipulability of a body of knowledge. (Bruner, 1966a, p. 41)

According to Bruner, the structure of a subject influences a student's ability to learn, remember, use, and reason intuitively about mathematical ideas.

(1) Learning:

> Good teaching that emphasizes the structure of a subject is probably even more valuable for the less able student than for the gifted one, for it is the former rather than the latter who is most easily thrown off the track by poor teaching. (Bruner, 1960, p. 9)

(2) Remembering:

> Perhaps the most basic thing that can be said about human memory, after a century of intensive research, is that unless detail is placed into a structured pattern, it is rapidly forgotten....knowledge one has acquired without sufficient structure to tie it together is knowledge that is likely to be forgotten. (Bruner, 1960, pp. 24-30).

(3) Using:

> The continuity of learning that is produced by...transfer of principles is dependent upon mastery of the structure of the subject matter....That is to say, in order for a person to be able to recognize the applicability or inapplicability of an idea to a new situation and to broaden his learning thereby, he must have clearly in mind the general nature of the phenomenon with which he is dealing. The more fundamental or basic is the idea he has learned, almost by definition, the greater will be its breadth of applicability to new problems. (Bruner, 1960, p. 18)

(4) Intuiting:

> Usually intuitive thinking rests on familiarity with the domain of knowledge involved and with its structure, which makes it possible for the thinker to leap about, skipping steps and employing short cuts in a manner that requires a later rechecking of conclusions by more analytic means...Intuition consists in using a limited set of cues, because the thinker knows what things are structurally related to what other things. (Bruner, 1960, pp. 58-62)

Many of the above ideas about the role of structure were discussed with respect to Gagné's theory in Part II of this paper. Although Bruner and Gagné do arrive at similar conclusions about the importance of structure, their treatments of structure differ significantly. For Gagné, learning the structure of the subject matter was not itself an objective of instruction; whereas for Bruner, it is. Furthermore, while Gagné focused almost exclusively on the influence of lower-order prerequisites on higher order ideas, Bruner addresses interrelationships among ideas at various levels. Bruner is also more concerned about the intuitive mastery of whole systems of ideas, not just the formal learning of isolated concepts or rules.

Although Bruner's treatment of structure is quite different from that of Gagné, most of the important structure-related abilities suggested by his theory are also involved in his discussion of the role of language and symbolism. Therefore, this section will focus on abilities related to the use of language and symbolism in mathematics.

## The Use of Disciplinary Language and Symbolism

According to Bruner (1966a), "Man's use of mind is dependent upon his ability to develop and use 'tools' or 'instruments' or 'technologies' that make it possible for him to express and amplify his powers" (p. 24). That is, through the use of cultural amplifiers, "cognitive growth might be conceived as achieving a capacity for simplicity in dealing with information" (Bruner, 1966b, p. xi).

Among the most powerful "cultural amplifiers" civilization has developed are the various disciplines of knowledge. For example, mathematics has evolved to a state in which many of its most basic truths are expressed in language that sounds deceivingly simple and true to non-mathematicians. In beginning calculus courses, students often wonder why facts like "the integral of the sum of two functions is equal to the sum of the integrals" are worth proving. When ideas such as these are stated in clever ways, they often seem obvious--disguising a number of subtle but important theoretical issues.

Halmos (1958, p. 67-69 ) writes:

> Mathematics has grown so luxuriantly in the past 2,000 years that it must be continually polished, simplified, systematized, unified and condensed. Otherwise the problem of handing the torch to each new generation would become completely unmanageable. No man alive today can know, even sketchily, all the mathematics published in the last 10 years.
>
> After a couple of centuries 10 of the greatest discoveries of the era are likely to find themselves together between the covers of a slim volume in the pocket of a graduate student who, with luck, will absorb them all in two or three months.
>
> The organizational system that a discipline uses, and the language and symbolism it develops are very useful "for simplifying information, for generating new propositions, and for increasing the manipulability of a body of knowledge." Some of the most important functions that a discipline performs are to select and organize ideas that are most important and basic, and to interpret and represent these ideas in a form that will be useful for dealing with applied problems and for laying the groundwork for continued development of new ideas.

The refined language and symbolism of mathematics plays an important role in both the generation of new ideas and the learning of old ideas. For example, in their pioneering work in psycholinguistics, Miller and Chomsky (1963, p. 488) concluded that "sentences have a compelling power to control both thought and language," and in no subject matter area is this statement more true than in mathematics.

Bruner (1966b) writes:

> ...Having translated or encoded a set of events into a rule-bound symbolic system, a human being is then able to transform that

> representation into an altered version that may but does not necessarily correspond to some possible set of events. It is this form of effective productivity that makes symbolic representation such a powerful tool for thinking or problem solving. (p. 37)

However, verbal precociousness does not always affect the behavior of children or adults.

> ...Once we have coded experience in language we can (but not necessarily do) read surplus meaning into the experience by pursuing the built-in implications of the rules of language...there is some need for the preparation of experience and mental operations before language can be used. (Bruner, 1966b, p. 51)

> The syntactical maturity of a five-year-old seems unconnected with his ability in other spheres, He can muster words and sentences with a swift and sure grasp of highly abstract rules, but he cannot, in a corresponding fashion organize the things words and sentences "stand for"....In order for the child to use language as an instrument of thought, he must first bring the world of experience under the control of principles of organization that are in some degree isomorphic with the structural principles of syntax. Without special training in the symbolic representation of experience, the child grows to adulthood still depending in large measure on the enactive and ikonic modes of representing and organizing the word, no matter what language he speaks. (Bruner, 1966b, p. 47)

The above comments underscore the fact that issues concerning mathematical language and symbolism are closely related to issues concerning the organization and structure of the ideas the language describes. The mutual interdependence between structure and language are emphasized even more clearly in Bruner's introduction to Dienes' book, An Experimental Study of Mathematics Learning (1963).

> The symbols of a language--a natural or a mathematical language--can either be viewed as transparent or opaque. When we treat symbols as transparent we are principally mindful of the referential function, what they "stand for" or "mean." But it is also possible to treat a symbol system without regard to what lies beyond the symbols in the world of experience, to treat the system as a self-sufficient body of rules for forming and transforming sentences or equations or functions. (p. xi)

In a mathematical system, part (or all) of the meaning of the word comes from the system in which it is embedded. For example, in an axiomatic system, the meanings of the undefined terms can be considered to derive entirely from their relationships with other undefined terms.

Of course, when mathematicians do mathematics, undefined terms usually are assigned more meaning than the purely syntactical meaning that is derived from their axiomatic descriptions. That is, a "model" (or a series of models) is used to give additional meaning to the undefined terms and to verify the consistency of the axiomatic system. But, in a simple sense, the modeling processes that mathematicians use consist of matching an idea within one representational mode with the corresponding idea within another representational mode. Therefore, a given idea may acquire meaning from at least three types of relational systems: 1) between-idea structures within a given mode of representation of the type discussed by Gagné, 2) within-idea systems of the type discussed by Piaget, and 3) between-mode structures that relate ideas in one mode to corresponding ideas in another mode. Using this last type of relational system requires processes that involve translating from one mode of representation to another. These "translation" processes are another source of potential abilities and/or disabilities in mathematics.

## The Use of Various Modes of Representation

When Bruner discussed the processes he believes are involved in mastering cultural amplifiers pertaining to a subject matter area like mathematics, he particularly focused on conceptual mechanisms that move a child from enactive to ikonic to symbolic modes of representation. For example, in Toward a Theory of Instruction (1968), Bruner writes:

> The new models are formed in increasingly powerful representational systems. It is this that leads me to think that the heart of the educational process consists of providing aids and dialogues for translating experience into more powerful systems of notation and ordering. (p. 21)
>
> In mathematics, cognitive growth can be characterized by a transition to progressively more powerful and economical representation systems. There is serious disequilibrium when two systems of representation do not correspond--what one sees with how one says it, or how one must act overtly and how the world appears. Indeed...it is usually when systems of representation come into conflict or contradiction that the child makes sharp revisions in his way of solving problems... (1966b, p. 41)

For Bruner, some of the most important mathematical abilities are likely to be related to processes that are involved in moving from one mode of representation to another. These "translation" abilities will be the focus of this section.

In mathematics, these between-mode translation processes are particularly interesting because: (1) there usually exist several independent written-symbolic ways of representing a given mathematical idea (e.g., we can describe the ideas using normal English sentences or we can use mathematical symbols); (2) there usually exist several independent spoken-symbolic ways of describing an idea (e.g., in mathematics many words like "and," "or," "if...then," "add," "multiply," etc. are given meaning that do not correspond directly to their usual everyday meanings; (3) written-symbolic representations and spoken-symbolic representations often involve subtle differences that can create or conceal confusions (e.g., why do we say "eleven" for the numeral 11, rather than saying "tenty-one"--to be consistent with twenty-one, thirty-one, etc?)

## The Ability to Translate from One Mode of Representation to Another

This section will focus on processes that are involved as a student moves back and forth from enactive to ikonic to symbolic modes of representation for a given idea or system of ideas. In the enactive mode the world is known principally by the habitual actions that are used for coping with it. The ikonic mode of representation uses imagery that is often relatively free of action and is based on figurative or perceptual properties. Finally, the symbolic mode translates actions and images into written or spoken language.

There are a variety of ways to make an idea meaningful--some of which involve "translation" processes corresponding to the arrows in Figure 7. These processes not only play an important role in the development of mathematical concepts, they are also among the most important "modeling" processes students use when they try to apply the concepts in real life situations.

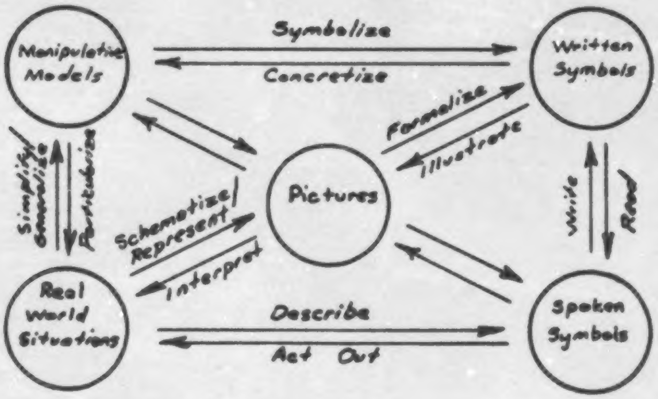


Figure 7

In Figure 7, ikonic representations are partitioned into two subcategories: (a) pictures and (b) manipulative materials (like Cuisenaire rods, arithmetic blocks, or counting discs). Similarly, symbolic representations are partitioned into two subcategories: (a) written symbols and (b) spoken symbols. The designation of five categories in Figure 7 does not mean that the categories are completely distinct from one another. For example, distinctions between a real-world situation and a manipulative model result from the fact that the models usually involve less "noise" (i.e., attributes that are irrelevant to the concept they are intended to embody) and the models are usually used in a symbolic way to represent many different real-world situations. Similarly, distinctions between a manipulative model and a picture result from the facts that pictures are often more abstract (e.g., they represent notions with arrows or suggested activity; they represent three dimensional objects in two dimensions) and pictures usually are not intended to be manipulated.

The five categories in Figure 7 also are not intended to suggest monolythic representational modes. That is, we do not have just one single written symbolic mode, one single spoken symbolic mode, or one single manipulative mode. There are a variety of semiautonomous subsystems within each of these modes. For example in the written symbolic mode, a basic definition from calculus can be written using an ordinary English sentence or it can be written in the form shown below.

$$\text{Def: } \lim_{x \to a} f(x) = L \quad \mathit{iff} \ \forall \varepsilon > 0 \ \exists \delta > 0 \rightarrow: |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon$$

A simpler example to illustrate the above point could involve any simple arithmetic equation. For example, 1/2 x 1/3 = 1/6 may be written in any of the following forms (among others):

One-half of one-third is one-sixth.
One-half multiplied by one-third equals one-sixth.
One-half times one-third equals one-sixth.

The above example also suggests how a variety of spoken-symbolic representations can be used to express a single mathematical idea. Similarly, Part III of this paper gave examples to show how a variety of different manipulative models can be used to represent a single idea. Therefore, in addition to the "between mode" translation processes that are illustrated in Figure 7, there also exist important "within mode" translation processes. Some of these written-mode translation processes were discussed in Part III. Other within-mode translation processes (e.g., symbols to symbols) are also important but will not be discussed here and are not represented in Figure 7.

One final disclaimer should be made concerning the representational modes and translation processes in Figure 7. That is, the names given to the translation processes are simply convenient titles to be used for future reference within this paper. They are not intended to imply, for example, that generalizing is the process of translating from real-world situations to concrete models. The various processes could have been labeled P1, P2, P3, etc. But the risk involved in using meaningful names is minimal as long as they are not interpreted as final definitions.

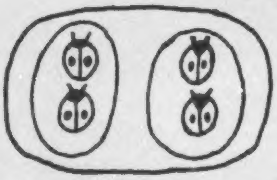
The examples below illustrate difficulties related to the between-mode translation process in Figure 7.

(1) Concretizing: Written Symbols to Manipulative Models

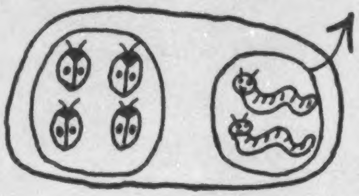
Susan (second grade): Susan had been taught basic addition facts and subsequent "harder" problems using a counting frame abacus. Her teacher noticed that Susan seemed stimulus bound and depended heavily on this particular concrete aid. Her teacher wanted to make Susan more independent of her abacus so she could rely on the underlying ideas she had learned rather than on the manipulative aid. The teacher decided to have Susan use a new embodiment, Cuisenaire rods, in order to wean her away from her dependence on any one particular aid. But, Susan was not able to do computation with Cuisenaire rods. She continually tried to relate the rods to her abacus, and she became quite confused in the process. In Susan's case, difficulties in translating from written symbols to a manipulative model were closely related to difficulties in translating from one model to another.

(2) Representing: Manipulative Models to Pictures

Cheryl (second grade): Cheryl's teacher noticed that Cheryl had difficulty understanding the illustrations given in class or the addition pictures in her book. Cheryl had far more success when poker chips or other manipulative aids were used to illustrate written addition problems. Cheryls' teacher reasoned that if Cheryl could translate from written symbols to a manipulative model, but not from written symbols to a picture, then perhaps Cheryl could be helped if she practiced translating from manipulative models to pictures.



$2 + 2 = 4$



$6 - 2 = 4$

Unfortunately, Cheryl also had great difficulty translating from poker chips to set-like pictures. She could perform addition operations using real chips but she could not coordinate the actions needed to draw pictures of the illustrations. She also had difficulty understanding how to use static pictures to represent the activity of addition. Some of Cheryl's other difficulties are described below.

(3) Concretizing: Pictures to Manipulative Models

Cheryl (same as above): Cheryl's teacher asked her to build some clay-and-toothpick models of simple 3-dimensional shapes. Pictures like Figure 8 (below) were used as models. The pictures were photographs of real clay-and-toothpick shapes. But, the models Cheryl built were two-dimensional, not three-dimensional; that is, Cheryl's model looked like a square with an X inside (see Figure 9); she used five balls of clay rather than four, and she used eight toothpicks rather than four.

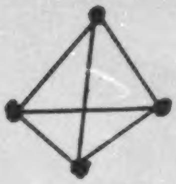


Figure 8

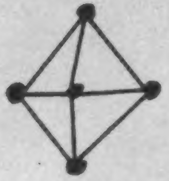


Figure 9

In a similar situation, Cheryl was asked to use modeling clay to make shapes like those shown in photographs in her book. Again, Cheryl had great difficulty doing these tasks correctly. Even after some extremely creative attempts by her teacher to explain the task in a meaningful and understandable way, Cheryl rolled out a flat sheet of clay and drew lines on the clay that were as much like the photographs as she was able to do. She was unable to make a 3-dimensional clay figure like the 2-dimensional figures in her book. In fact, she had great difficulty with nearly any task that required her to translate from pictures to manipulative objects or from manipulative objects to pictures.

(4) Describing: Spoken Words to Manipulative Models

Steve (fourth grade): Steve was excellent in every subject except arithmetic. But he also had great difficulty understanding spatial relationships. One day his special education teacher put Steve on one side of a table and one of his friends on the other side (see Figure 10). Then she put a partition between the two boys so that they could not see one another. Next, she gave Steve's friend a stack of poker chips and put a 3 x 3 array of poker chips in front of Steve. She asked Steve to describe the pattern so that his friend could copy it exactly. Steve was unable to do this. He was also unable to play similar "communication" games involving geoboard shapes, logo blocks, or "treasure hunt" maps.

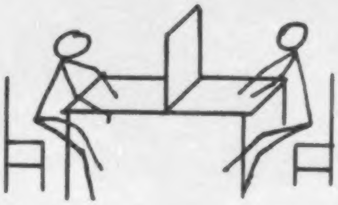


Figure 10

Because Steve had so much difficulty describing sets of objects and geometric shapes, his teacher decided to let Steve switch roles with his friend. That is, his friend became the "describer" and Steve was the "maker." Again, however, Steve was unable to perform simple tasks of this type correctly. He had great difficulty translating from spoken words to manipulative models--or vice versa.

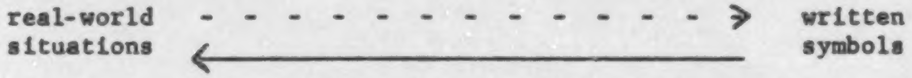
### (5) Interpretation: Written Symbols to Real Situations

Kevin (sixth grade): Kevin nearly always got high scores on computation tests in arithmetic. But he seemed unable to solve the simple story problems in his book. His LD teacher walked around with him one day pointing out real situations (like those described in his book) in which she thought Kevin's arithmetic skills could be used. She was surprised to find that Kevin was seldom able to select the appropriate arithmetic operation to describe the situations she identified. Also, if she gave Kevin an equation like $12 \times 7 = \square$ he was usually unable to find real situations that the problem could be used to describe. Kevin was unable to relate his written-symbol understanding to real situations.

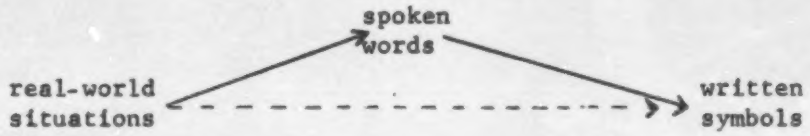
The preceding examples do not exhaust all of the processes that are represented in Figure 7, but they do illustrate some of the types of difficulties that can occur concerning between-mode translation processes. They also suggest a number of tasks for diagnosing translating difficulties and a number of types of remedial activities.

To generate a whole series of diagnostic tasks for a given idea, the teacher can present the idea in one mode and ask the student to illustrate (or describe, or represent) the same idea in another mode.

If a student has difficulty with some particular between-mode translation process, remedial activities can often be generated by practicing the inverse of the difficult process. For example, a child who has difficulty translating from real-world situations to written symbols may find it helpful to practice translating from written symbols to real-world situations.



Or, difficult processes can be broken up into a series of easier processes. For example, a child who has difficulty translating from real situations to written symbols may find it helpful to begin by translating from a real situations to spoken words and then from spoken words to written symbols.



The processes in Figure 7 are important for a variety of reasons.

(a) They are simplified versions of the "modeling" processes used by gifted applied problem solvers. They represent some of the most important processes students need when they try to use basic geometric, algebraic, or number concepts.

(b) When we say a student "understands" a mathematical concept, part of what we mean is that he/she can use the kind of processes listed in Figure 7. Yet, students are given few instructional activities that focus directly on these processes--in spite of the fact they are the kind of processes that give meaning to the ideas teachers are trying to teach.

(c) Average or below average students can learn to use these processes. Yet, work with special education students indicates these problems cause difficulties for many students--and that these difficulties can severely restrict problem solving (or even concept formation) capabilities.

(d) Teachers do not need to wait for large-scale curriculum projects to develop special instructional activities to teach these processes. They can be built into the kind of lessons that are included in many textbooks and the kind of problems that are included in the "applied" sections of national assessment tests.

(e) If diagnostic questions indicate a student is having unusual difficulties with one of the processes in Figure 7, other processes in the diagram can be used to strengthen or bypass the difficulty.

## Some Other Types of Abilities

In addition to the between-mode translation processes that are given in Figure 7, within-mode translation processes can also be sources of difficulties. For example, children may have difficulty translating from one written-symbolic statement to equivalent written-symbolic statements, from one verbal description to an equivalent verbal description, or from one picture to another. Or earlier, sections of this paper gave examples of children who have difficulty translating from one manipulative model to another. These sorts of within-mode translation processes often seem to function as important prerequisites for certain between-mode processes. For example, a child who has difficulty translating from manipulative models to written symbols may find it helpful to practice translating from one manipulative model to another. However, the nature of within-mode translation processes are similar to the between-mode processes that have been discussed here. Therefore, without depricating their importance, no further discussion will be given about them in this paper.

Bruner's emphasis on language and symbolism also suggests the possibility that mathematics involves some unique reading and language processes that are related to underlying mathematical abilities. But, at the present time, the kinds of mathematics reading processes that seem to be the most likely candidates to be related to mathematics abilities are the kinds of within- and between-mode translation processes that have already been discussed in this paper.

## Some Final Comments About Bruner's Perspective

Bruner's "enactive to ikonic to symbolic" description of cognitive growth seems similar to Piaget's "preoperational to concrete operational to formal operational" description. However, these two dimensions of cognitive development are quite distinct. Piaget's stages are defined by the within-idea operational complexity of the ideas a student is able to use, whereas Bruner tends to de-emphasize the importance of within-idea structures and instead emphasizes between-idea systems and the role of language and written symbols. Bruner's stages have to do with the mode of representation that is used to describe an idea and its relationship to other ideas.

## Concluding Remarks

The examples that were given in this paper were based on extensive interviews (i.e., lasting at least one hour) with more than seventy children who had been identified by their school districts as having a mathematical learning disability. The examples should not be considered to be "typical" LD children. In fact, it is not clear what a typical LD child would be. In many schools, unfortunately, LD is simply a convenient label for problem children that their teachers do not know how to deal with effectively.

All of the children who were mentioned in this paper came to the attention of the author because of graduate students, former students, local LD teachers, and colleagues who invited the author to interview some of their interesting LD cases. The students who were cited as examples were children who were identified as "particularly interesting" to a person whose interests are in mathematics learning. This population represents considerably less than half of the LD children in most schools.

The purpose of this paper was not to form generalizations about all LD children. Instead, it was to identify some important mathematical abilities that LD children (or normal children who are having difficulty in mathematics) may lack. In fact, because research in this area is in an extremely primitive state, the primary objective was to help researchers and practitioners ask better questions about possible mathematical learning disabilities.

The most striking characteristics of mathematics have to do with its structure and its distinctive use of language and symbolism. Therefore, if there exist any abilities that are unique to mathematics (in the sense that they are not just specific manifestations of general intelligence), one should expect that they may be related to these distinctive characteristics. This paper has discussed some possible mathematical abilities from the point of view of three theorists who emphasize three different kinds of structures. Gagné emphasized between-idea structures; Piaget emphasized within-idea structures; and Bruner emphasized between-mode structures.

One of the most obvious characteristics of mathematically gifted students is that they see relationships among things that normal children do not understand to be related. Similarly, one of the striking characteristics of LD students who are having difficulty in mathematics is that they do not see relationships among things that we assume most normal children do understand to be related. Some of these relationships have to do with between-idea structures, some have to do with within-idea structures, and some have to do with between-mode structures. But, in all cases, deficient structural relationships can cause striking difficulties in the mathematical reasoning of children.

It seems reasonable to expect that many of the processes and abilities that were discussed in this paper may also be deficient in many normal children. Certainly, little instructional time is devoted to helping children acquire most of the abilities and processes that were mentioned. Research in this area should significantly increase the mathematical learning and problem solving capabilities of most students.

How are the abilities and processes in this paper related to central processing dysfunctions? Answers to these sorts of questions will demand better descriptions of the kinds of abilities and processes that may be effected. Hopefully, the ideas presented in this paper will be helpful in identifying abilities and processes that are critically important in mathematics learning.

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# Information Processing Analyses of Mathematical Problem Solving

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A growing body of psychological theory has focussed on the processes and structures involved in human cognition. Information-processing analyses have been performed and models of performance developed at increasingly sophisticated levels of specificity. An understanding of general problem solving has been emerging from these efforts, while analyses of problem solving in specific scientific and mathematical domains have contributed a number of additional concepts. The objective of this chapter is to summarize current understanding of the psychology of mathematical problem solving by reviewing these bodies of knowledge and synthesizing concepts from general problem-solving theory with those from the technical problem-solving literature.

## General Problem-Solving Theory

Interest in human cognitive processes and representation of knowledge structures can be traced back to early European psychologists such as Selz (1913, 1924) and Bartless (1932). Gestalt psychologists, including Duncker (1945), Köhler (1927), and Wertheimer (1945/1959), stressed the importance of understanding in achievement of problem solutions. These Gestalt theories emphasized the insightful nature of problem solving. They explained problem solution as involving a sudden understanding of a situation and an integration of previously learned responses in a novel way.

A substantive theory of the ways in which humans process information to arrive at problem solutions emerged with the work of Newell and Simon (1972). They conceptualized problem solution as the successful outcome of search processes, and provided a language, in effect, for expressing and operationalizing concepts central to a cognitive theory of problem solving.

Newell and Simon's theory is based on extensive analyses of adults solving short, well-defined problems of a symbolic nature. Although the three main tasks used--chess, symbolic logic, and cryptarithmetic--comprise a narrow sample of problem types, the theory is proposed to account for a broad scope of behavior. Central to Newell and Simon's approach is the assertion that humans, when engaged in problem-solving behavior, can be characterized as information-processing systems (IPS's). As a theoretical construct, an IPS is proposed to account for the basic mechanisms of cognition. Some of the strong, central assumptions about the components of cognition include the presence of long-term and short-term memories (LTM and STM), and processes of retrieval and storage in memory, pattern recognition, comparison processes, and symbol manipulation. Explanations of human problem-solving behavior at this level of detail have been found to be amenable to computer simulation. A theory of human problem solving behavior can therefore be expressed as a program on another IPS (a computer), and the adequacy of the program for modeling human behavior can be evaluated through comparison of the two systems' performances. This is especially true because an information-processing theory is dynamic--i.e., implemented as a computer program, the theory describes in detail the changes in a system's knowledge state and sequences of processes through time. Computer traces are typically compared with protocols of human subjects "thinking aloud," resulting in validation or modification of the theory tested.

Newell and Simon constructed programs and tested them against human performance in the manner described above. Because human performance could be adequately accounted for empirically in terms of mechanisms like those in their programs, the resulting description of problem solving processes were considered characteristic of any problem solving IPS, including humans. The characteristics of problem solving here described are those which Newell and Simon assert to be invariant over task and problem solver.

All problem solving is posited to take place in a problem space. Newell and Simon assert that a problem solver must construct and work within an internal representation of the task environment; the problem space contains not only the actual solution but all possible solutions the problem solver might consider. Presented with a particular problem, the individual must encode the components of the problem in a space consisting of the initial as well as desired goal situation, a set of elements representing intermediate knowledge states, a set of operators, or information processes, that may be applied to produce new states of knowledge from existing states, and all other available concepts or relations needed to understand these situations. Thus, this problem space includes the bases for all overt behaviors eventually exhibited by the problem solver as well as behaviors only considered in thinking about the problem. For example, in a chess player's problem space the elements or knowledge states are all possible chess positions; the initial state is the starting board position; the final state is indefinite but is characterized by attainment of a checkmate situation; the operators are legal moves, etc. In cryptarithmetic (a puzzle requiring assignment of distinct digits to letters such that, for example, the statement "DONALD + GERALD = ROBERT," would be arithmetically true, given that D is 5), the elements are all possible combinations of digit-letter assignments; the initial state is the configuration given, with the knowledge that D is 5; the goal situation is not known explicitly but consists of assignment of digits to all letters such that the problem is arithmetically possible; operators are the processes for making all possible assignments; other relations or concepts needed might include parity, equality, and inequality.

Problem solving is postulated to take place by search in a problem space. This statement must be understood in conjunction with the fact that other behaviors relevant to problem solving may be exhibited as well--i.e., when a problem is presented, it must be recognized and understood, a problem space must be constructed or evoked from LTM, and problem spaces can be changed during the solution process. These aspects of solution are not themselves searches in a problem space; they are crucial contributory activities but final solution is achieved through a search process. Search consists of a consideration of one knowledge state after another until a desired knowledge state is reached--that is, problem solving is concerned with finding a path through the space from initial problem state to goal state. Movement to new knowledge states is accomplished by analysis of the current problem situation, and selection and execution of operators to produce new states. Search sometimes involves backup--return to old knowledge states and abandonment of some knowledge state information. This occurs when a dead-end or contradiction is reached (e.g., realization in cryptarithmetic that a required digit-letter assignment is impossible because digit has already been assigned).

Means-end analysis and goal-directedness are major characteristics of problem solving activity. A problem solver proceeds toward problem solution in a recursive, step-wise fashion, working to select means (operators) to achieve ends (goals). Use of means-end analysis involves comparison of the current knowledge state with the desired goal state to detect differences. Having recognized a difference, the problem solver sets a goal, taking action to remove the difference. An operator or transformation rule is sought to transform the current status. Using a table of connections that indicates which operations are relevant to which differences, the system either finds an operator that accomplishes the desired transformation, or sets a subgoal to reduce a high priority difference in a way that moves the state closer to the goal. Through a series of such transformations, the problem eventually reaches the point where application of an operator removes the remaining difference between existing state and goal state, and the problem is solved.

Planning processes are used to construct proposed solutions in general terms before working on detailed solution steps. Means-end analysis is limited by its mechanism of seeing only one step ahead at any given point. Planning counters this limitation, allowing the problem solver to explore the usefulness of a general strategy for reaching the desired goal state. As described by Newell and Simon, the planning method consists of:

1. Abstracting by omitting certain details of the original objects and operators.

2. Forming the corresponding problem in the abstract problem space.

3. When the abstract problem has been solved, using its solution to provide a plan for solving the original problem.

4. Translating the plan back into the original problem space and executing it. (Newell & Simon, 1972, p. 429)

Thus, the system explores the feasibility of solution plans by identifying general features of the problem situation and goal, and considering the general kinds of changes producible by available operators.

A further analysis of planning in problem solving was given by Sacerdoti (1975). In Sacerdoti's system, called NOAH (for network of action hierarchies), there is knowledge stored about actions that occur at different levels. For example, there is knowledge about the action of moving an object, which is known to include component actions of picking up the object, walking to another location, and setting the object down. For each action at whatever level, NOAH has knowledge of prerequisite conditions that must be present for the action to be performed, and consequences that result from performing the action. Stored knowledge of prerequisites and consequences enables powerful planning procedures, in which a sequence of actions can be planned so that prerequisites needed for future actions are considered in deciding which other actions will be included in earlier parts of the sequence.

The central concepts outlined above have been the subjects of subsequent theoretical and empirical examination. Expansion and further specification of the concepts have resulted and will be discussed in later sections of this chapter.

Newell and Simon's work provided the most thorough, detailed account of human problem solving yet achieved. However, there remained a need for strong theoretical development at the level of general psychological principles that explain performance in broad classes of problems. In an effort toward developing a coherent, general theory of human problem solving, Greeno (1978) outlined a set of features and processes required in problem solution. His effort to conceptualize problem solving concepts in a general theoretical framework can be utilized to examine the applicability of general problem solving theory to the domain of mathematical problem solving.

Based on hypotheses about the general kinds of psychological skills required for problem solution, Greeno suggested a typology of problems comprised of three ideal types. Problems of inducing structure were described as those for which some elements are given and the task requires identification of the pattern of relations among the elements (e.g., analogies, series extrapolation). Greeno proposed that the main cognitive ability required for solving problems of this type is a form of understanding, i.e., apprehending relations and constructing an integrated representation. Problems of transformation involve an initial situation, a goal, and a set of operations for producing changes in situations (i.e., move problems such as Tower of Hanoi, and change problems including theorem proofs). Solution of transformation problems relies primarily upon use of means-end analysis and planning processes. Tasks referred to as arrangement problems are those that present some elements and require the problem solver to arrange them in a way that satisfies some criterion (e.g., anagrams, cryptarithmetic). Solution requires a process of constructive search, requiring generation of the possible knowledge states that constitute the search space, and search for the solution in that space. Greeno asserts that most problems are not pure cases of one problem type, but rather include strong components of the three types and involve corresponding demands for the various problem solving processes.

The remainder of this chapter will include discussion of the knowledge and processes required for solving problems in mathematics, and for solving practical problems in which mathematical knowledge is required. The latter discussion will focus on solution of word problems, where studies of the processes have been conducted. Our goal in this discussion is to use the general concepts that have been developed in the theory of problem solving to provide an integrative framework for discussion of the available literature on solution of technical problems that are relevant to mathematics instruction. First, we will discuss problem solving processes that are required for solving ordinary exercises of the kind that are used as homework and test problems. The final section will present analyses of problem solving involving applications of mathematical knowledge.

## Mathematical Problem Solving

Most mathematical instruction consists of training for solving specific kinds of problems, and the activity that students engage in as they do exercises or take tests consists of problem solving processes. Although most mathematics educators consider "problem solving" to refer only to relatively unusual parts of the curriculum (thus feeling that there should be more attention to problem solving in the curriculum), this view does not take into account what it is that students must actually do in order to succeed in the ordinary tasks that constitute most of the curriculum.

The claim that ordinary exercises require problem solving is consistent with many persons' intuitions about geometry. Consider the problem: "Find the length of a diagonal of a rectangle whose perimeter is 20 in. and whose width is 4 in.," (Jurgenson, Donnelly, & Dolciani, 1972, p. 323). This has the structure of a transformation problem. There is an initial situation, consisting of given information, and there is a goal of finding a quantity that is not provided but that can be inferred from the given quantities. Problem solving operators involve rules for inferring quantities from other quantities, for example, inferring the length of a rectangle from its perimeter and width. The problem is solved when a sequence of operations leads to an inference that assigns a quantity to the diagonal. Another example is the problem, "Prove: If the diagonals of a parallelogram are congruent, the parallelogram is a rectangle," (Jurgenson, Donnelly, & Dolciani, 1972, p. 243). This problem has an initial situation consisting of some statements, with the goal of forming a proof of one of the statements. This is an arrangement problem, where the student must construct a sequence of statements that are related to each other according to the rules of proof. The structure that is to be produced starts with given information and consists of a series of inferences that terminates with the statement to be proven. Thus, standard proof problems in geometry have components of both arrangement and transformation problems, since the structure that students must arrange is a series of transformations.

Standard exercises in algebra also have the defining features of problem situations. The following is an example: "Solve the following compound sentences: $5x + y = 15$ and $3x + 2y = 9$ ," (Payne, Zamboni, & Lankford, 1969, p. 230). The initial situation is the pair of equations that is given. The goal is a different pair of equations that are to be derived, assigning values to $x$ and $y$ . Each step in the solution involves a transformation of an equation that is given or that has been derived, or a derivation of a new equation from two earlier equations. Another example is the problem of simplifying an expression, such as (Payne, Zamboni, & Lankford, 1969, p. 394):

$$\frac{x^2 + 8x + 16}{x^2 - 9} \cdot \frac{x - 3}{x + 4}$$

Problems of simplifying expressions are also problems of transformation, but the goal is not well defined. The given expression is the initial situation. Problem solving operators are the rules for factoring, cancelling, and so on. The problem is solved when an expression is found for which no further problem solving operators can be applied.

The claim that mathematical exercises require procedures of the kind needed for problem solving also holds for standard computational problems. Although some readers may be unconvinced by the fact that even a simple computational exercise presents an initial situation and a goal that must be reached by a sequence of operations, it is harder to remain skeptical after seeing the results of a serious attempt to analyze the procedural knowledge required for a system that performs simple column-addition and column-subtraction problems, much less the knowledge required for more complex operations such as those involved in fractions. Consider subtraction. The problem solver must know how to break the problem into subproblems, each with its subgoal of finding the correct number to place in each column of the answer. In each subproblem, a condition must be tested to determine whether the subtraction operation (top minus bottom number) can be performed. If it cannot (because the top number is smaller) an additional subgoal must be formed that is satisfied when the situation is transformed (by borrowing) so the subtraction operation can be performed. An analysis complete enough to provide for typical errors has been given by Brown and Burton (1977). The knowledge structure they hypothesized for the task of subtraction is shown in Figure 1. Its structure is that of a procedural network, a formalism developed by Sacerdoti (Note 1) to represent knowledge for problem solving somewhat more complex than that generally represented by production systems. While it is unusual to include "simple" computational exercises in the domain of "problem solving," it seems to us that this does a considerable disservice to young students who acquire very complex knowledge structures in order to perform these tasks, especially when analysis of their knowledge reveals that its structural features are as complex as any that have been developed in the theory of problem solving.

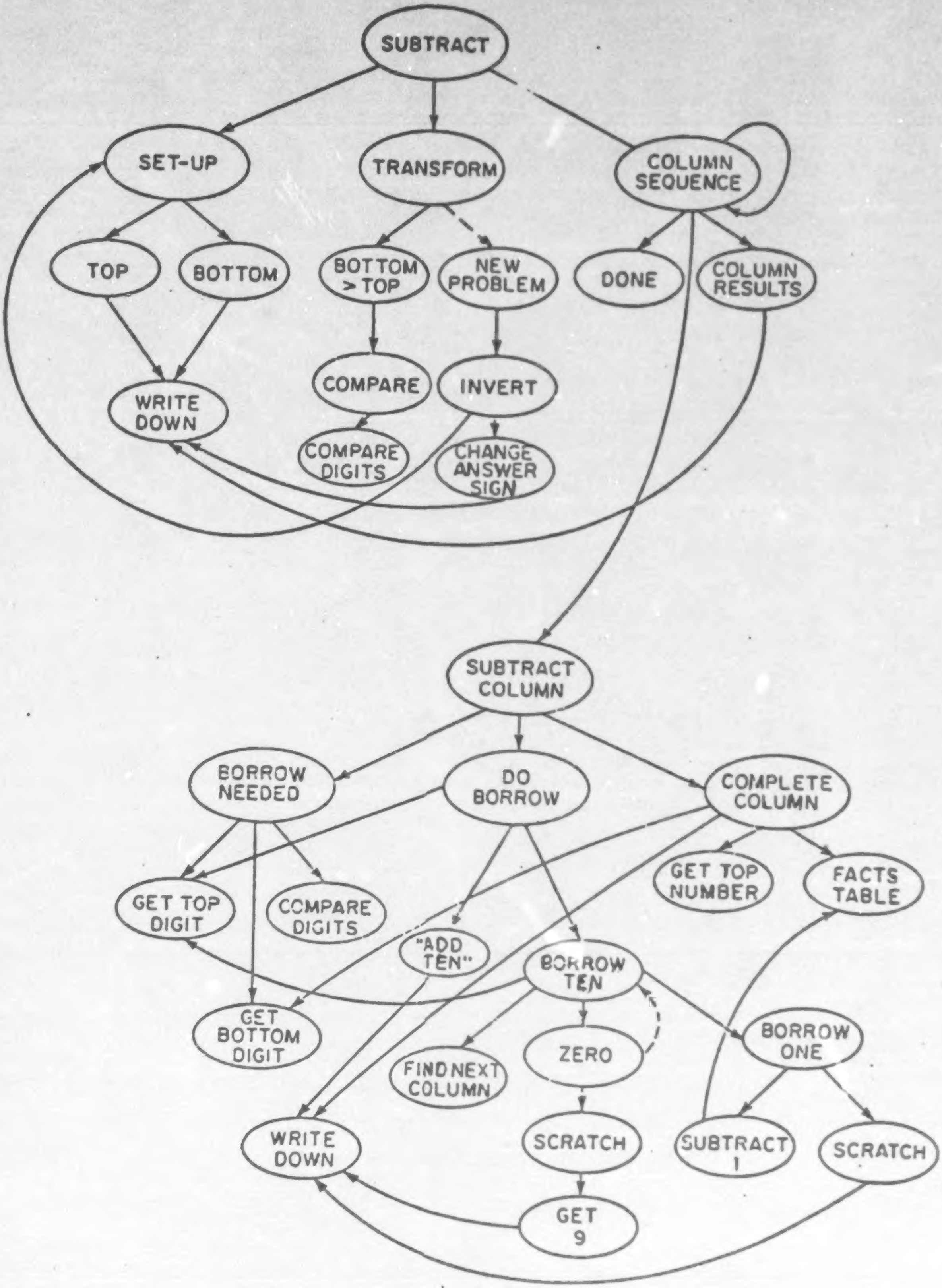

Figure 1

A Procedural Network for Subtraction (Brown & Burton, 1977)

A preliminary analysis of strategies in solving algebraic equations has been given by Bundy (Note 2), and discussed by Brown, Collins, and Harris (1977). In Bundy's analysis, knowledge for solving equations includes a basic method, consisting of three strategies, and an auxiliary method for removing nasty function symbols. The strategies in the basic method are called isolation, collection, and attraction. Isolation involves operating on an expression so that the unknown term is not embedded in expressions for functions. Collection involves reducing the number of occurrences of the unknown variable. Attraction involves transformations that bring occurrences of the unknown term closer in the expression; for example, bringing two occurrences to the same side of the equation. Strategies for removing undesired function symbols include inversion, such as squaring both sides of an equation to remove a radical, and use of definitions. These strategies correspond to a set of subgoals that the problem solver adopts in order to make progress on the problem. Instruction in algebra generally focuses on the operations that are permitted in transforming expressions and equations. Bundy's analysis makes it clear that knowledge of algebraic transformations is not sufficient for solving equations. The student must also have strategies and methods for choosing operations that will be helpful in making progress toward a solution. These strategies and methods are of the same general kind as the strategic knowledge that is critical in any problem solving system.

An analysis of processes used in solving various geometry problems has been developed (Greeno, 1976, 1977, in press). A model, called Perdix, simulates the process of proving theorems and computing the measures of specified angles or segments given measures of other things in a diagram. These are the standard exercises of geometry. It is clear that the processes required to successfully complete these exercises are the kind that we characterize as problem solving processes. They include strategic knowledge for setting subgoals and planning, as well as knowledge for recognizing patterns and making inferences based on general propositions.

Some results of the analysis of geometry problem solving have provided relevant information for a theoretical question that has been of interest in the psychology of problem solving. The question involves processes involved in solving ill-structured problems, and whether principles that apply to solving well-structured problems also apply in situations where problems are ill-structured. This question arose because much of the theory of problem solving has been developed by analyzing problems in which the initial situation, the goal, and the permissable problem solving operators are all definitely specified. Since many problem situations lack specificity of one or more of these components, generality of analyses such as Newell and Simon's (1972) has been questioned, for example, by Reitman (1965).

Two aspects of ill-structured problems appear in ordinary geometry exercises. One aspect involves indefinite goals. When triangles are proven congruent, the pattern that is found (SSS, SAS, ASA, or whatever) is not specified in advance, nor are specific patterns usually considered and used as targets for search (Greeno, 1976). The second aspect involves incomplete specification of the problem space by the initial situation of the problem. Problems that require construction of an auxiliary line present less than a complete initial problem space, and the problem solver must enrich the initially presented material in order to find a solution (Greeno, in press).

The processes needed to solve problems with indefinite goals or constructions do go beyond those developed in the theory of well-structured problem solving. However, they seem quite compatible with the processes of means-end analysis and planning developed in that theory, and should be considered as extensions of the earlier theory, rather than as fundamental revisions of it. To account for use of indefinite goals we can postulate that problem solvers represent goals using a pattern recognition system, rather than as a single object or combination of features. A pattern recognition system is able to identify any of a set of alternative patterns, and thus can be used to determine whether any of several possible ways of achieving a goal has been reached (Greeno, 1976). To account for the occurrence of constructions, we can postulate planning knowledge of the kind analyzed by Sacerdoti (Note 1), in which the individual knows prerequisites for solving problems in certain general ways. For example, one plan for proving that angles are congruent is to consider triangles that have those angles as parts, and then prove that the triangles are congruent in a way that makes the angles corresponding parts of the triangles. The main prerequisite condition of this plan is the existence of triangles that contain the angles. If this prerequisite is not present, it can lead to a construction that produces the needed triangles by adding an auxiliary line to the diagram.

## Word Problem Solving

Word problems have been the subject of a number of information-processing analyses. These problems consist of narrative passages describing a situation in which quantities and quantitative relations are important. They require the solver to read and understand the passage, and select and apply mathematical operators or scientific principles to determine the value of one or more unknown quantities. Word problem contexts range from basic, elementary level arithmetic to situations involving high level algebraic relations or technical domains such as statics or thermodynamics in physics.

The major difference between solution of ordinary mathematical exercises and solution of word problems is that word problems require a process of understanding the situation. The understanding process has been the subject of considerable investigation in recent studies, including Anderson (1976), Norman and Rumelhart (1975), and Schank and Abelson (1977). These analyses have considered the process by which an understander constructs a representation of the information contained in text, including the relationships between concepts that are mentioned in the message. An important factor in understanding is the degree to which the understander fills in missing information based on the general knowledge that the understander has. Depending on the degree of inferential processing carried out by the understander, the representation that is achieved may be relatively superficial, with loose connections among the concepts in the text, or relatively coherent and tightly integrated, with a strong relational structure composed of implications that are drawn from the understander's knowledge.

The possibility of understanding situations in ways that differ in depth and coherence implies that in solving word problems there is an important trade-off between the amount of processing accomplished in the understanding of the problem and the amount of processing left for the problem solving operations. An individual who represents the problem situation in a relatively complete way might be expected to have information in that representation that is useful in guiding the selection of problem solving operators. A weaker representation would provide a situation in which more search and planning would be required.

We will review the major information-processing analyses of word problem solving, and then attempt to synthesize the findings in terms of general concepts of processing skills and knowledge organization across problem domains. A major conclusion of this review will be that skill in solving word problems depends strongly on the problem solver's ability to represent the problem situation in a way that facilitates the processes of search and planning for problem solving.

Algebra word problem solving. Paige and Simon (1966) analyzed algebra word problem solution by comparing the behavior of adult solvers with the strategy of Bobrow's (1968) STUDENT program. STUDENT was an artificial intelligence effort intended to demonstrate the capability of the computer to interpret natural language input and derive mathematical equations in algebra word problem solution. The program solves most problems by a direct translation process--the problem statement is interpreted phrase-by-phrase, using syntactic function tagging, substitution, and transformation rules to construct a mathematical expression representing the problem situation. For example, a statement such as "the number of customers Tom gets is twice the square of two tenths times the number of advertisements he runs" is transformed directly into an expression, e.g., "$x_1 = 2 * (.2 * x_2)^2$ ." This is accomplished by assigning a variable, "$x_1$" to the initial noun phrase, "the number of customers Tom gets;" interpreting "is" as "="; "twice" as "2 *"; "the square of" determines the exponent, 2; etc. While Paige and Simon found this direct translation strategy to correspond with much of their human solvers' behavior, they also discovered evidence for a second solution mode. A number of their subjects constructed "auxiliary representations" of problem situations, generally in the form of drawings representing the physical problem situations encountered. Those individuals who typically relied on semantic, substantive information in the solution process were considerably more successful at discovering problem incongruities than subjects who relied on direct, syntactic problem interpretation. That is, application of direct translation to certain "contradictory" problems in Paige and Simon's task set resulted in impossible solutions (such as negative values for the size of physical objects), or in transformation of the problem to a related, physically possible, situation. Individuals who realized that these problems were not solvable were generally those who included auxiliary cues from their knowledge of the world in the internal or external problem representations that they constructed. Paige and Simon concluded that problem solvers differ in their reliance upon syntactic versus semantic processing in problem solution, but that most solvers included some amount of semantic information in their solutions. Therefore, they suggested that a theory of skilled algebra word problem solution would have to incorporate the use of semantic knowledge in the problem understanding process to adequately account for human problem solving behavior.

Hinsley, Hayes, and Simon (1977) also distinguished between two modes of algebra word problem solution. They designated as a "text grammar" approach one that emphasizes the importance of formal structure and postulates distributed decision processes. The direct translation process is considered in this category because it involves no choice at the time of reading and makes little use of semantics. The second, "schema," approach emphasizes the importance of semantic knowledge and major decisions occurring early in the comprehension process. STUDENT incorporates this type of process involving choice of schema at the time of reading with respect to "age" problems only. If a problem is an "age" problem, as determined by the presence of particular phrases such as "years old" and "as old as," the problem is solved by inclusion of special heuristics. Hinsley, Hayes, and Simon postulated that, if this second approach were characteristic of human solution of algebra problems, then the availability and reliance upon problem schemata, or problem category information, would be empirically verifiable. In a series of experiments, they did establish that people can categorize problems into types, and that people have knowledge about each problem type that is used in formulating problems of that type for solution. Furthermore, Hinsley, Hayes, and Simon again found evidence of two different solution procedures used by subjects in extracting appropriate equations from problem texts. One procedure was characterized by reading of the entire problem before formulating any equations or noting any relationships explicitly. These solutions included use of problem category information and retrieval of schema that aided problem solution. The second procedure was the line-by-line, direct translation approach described previously. Some semantic knowledge was found to be incorporated in a portion of the solutions of this type, however. Hinsley, Hayes, and Simon's results supported the interpretation that the line-by-line procedure is a default process that is used if the problem is not successfully matched to one of the solver's available problem type schemata. That is, schemata may be important only in the formulation of problems in which the semantics of the cover story match the problem structure in an expected way, and the schema in question is currently available to the solver. This ability to perceive the underlying mathematical structures of word problems and to relate problems on the basis of the perceptions has been investigated by Krutetskii (1976) and Silver (Note 3). Both investigators found that there is a significant relationship between problem solving ability and the capacity to perceive accurately the formal structure of word problems. In fact, the tendency to sort problems according to mathematical structure was significantly positively correlated with numerous general verbal and mathematical ability measures. Conversely, students of lower ability were found to attend more to contextual and syntactic problem features than to underlying mathematical structure.

Technical word problem solving. Studies of solution processes in technical, scientific problem domains have focussed both on individual differences and the basic nature of processing.

After analyzing the physics problem solving behavior of one novice and one expert subject, Simon and Simon (in press) concluded that the expert's solution generally involved translation of the problem passages into physical representations, then use of those representations to select and instantiate the appropriate equations. They refer to this schema construction as "physical intuition" and constrast it with solving "simply by plugging in the formulas." The novice's behavior was typically "algebraic" in that she appeared to go directly from the problem statements to the equations required to solve them, with no evidence of any mediating cognitive representation. The novice's behavior was further characterized by explicit use of means-end analysis. She typically identified the goal and worked backward, setting up equations to solve for subgoals until the final solution could be reached. Only on particularly difficult problems did the expert rely on this type of problem solving strategy.

Larkin (Note 4) analyzed the protocols of physics problem solvers working on rather complex problems. She found evidence for an initial "qualitative analysis" performed before any equations were generated. Protocol data revealed that in the preliminary phase of problem solution, expert subjects constructed representations of the physical situation described in the problem. These representations were qualitatively elaborated by inclusion of supplementary information required for understanding the problem situation that was not included explicitly in the written statement.

Skilled solvers were found to retrieve from memory tentative "chunks" or "schemata" of associated physics principles for consideration, and to choose an aspect of the representation to which the chunk might be applied. Problem features are elaborated further if necessary in relation to the chunk being considered; the solver attempts to determine whether the tentatively chosen cluster of principles is applicable to the problem representation. When the match is bad, the chunk or problem aspect is discarded and a new combination considered. Upon finding a matching chunk and representation, the solver seeks a solution procedure. Larkin reported that expert solvers appear to have solution procedures associated with schemata, and that these procedures are prioritized with respect to their utility for solution of problems of different types. Some solvers were observed to evoke equations immediately upon recognizing a schema-problem match, apparently having spontaneously retrieved one highly appropriate solution procedure. Other individuals executed preliminary traces of several solution procedures in general terms to select the most applicable and promising procedure. Once a promising procedure was selected, all of these solvers evoked and instantiated equations to reach accurate solutions. Some solvers, however, executed no preliminary traces, reached dead-ends, and needed to try other procedures until one finally led to solution. These lower skill individuals, that is, skipped directly to equation building and manipulation without an intervening step of checking the usefulness of procedures in an abstract planning space. Alternatively, this behavior can be interpreted as indicating inefficient or no prioritization of solution procedures.

Thus, skilled problem solving apparently involves construction of a comprehensive problem representation as well as evokation of an appropriate theoretical schema or cluster of relevant physical principles for application. Solution procedures are associated with these chunks of knowledge, and instantiation of equations is executed with great efficiency. Evidence of storage of physical principles in related configurations was found only for experts; knowledge appears to be less well organized, and solution procedures less often associated with available schemata, in lower skill solvers.

Skilled physics problem solvers have been seen to be characterized by their translation of the verbal problem statement into an integrated, internal or external representation of the physical situation. This representation is elaborated by inclusion of auxiliary information necessary for understanding the problem, but not included in the natural language passage. Novak (1976) has demonstrated this translation process in a computer program, called ISAAC, that is able to read, understand, draw pictures of, and solve a set of physics problems stated in English. Problem statements are read, parsed, and interpreted semantically to construct an initial internal model of the objects and relations in the problem. Once the problem has been translated in this way, the solution is guided entirely by the internal representation, with no further reference to the problem statement. The initial model is interpreted in terms of canonical object types. That is, based on the system's knowledge of physics, objects in the problem are considered with regard to their functions in the problem--i.e., a person may be represented as a member of the canonical class of "pivot" when carrying a plank, or as a "point mass" when standing on one. This selection and completion of canonical object frames is considered by Novak to be one of the most important aspects of skilled physics problem solving ability. Once this interpretation has been accomplished, ISAAC constructs a geometric model of the problem, considering the location and orientation of objects in relation to one another. Although ISAAC proceeds to generate and solve equations at this point, it should be noted that human problem solvers would probably be observed to draw external representations of the internal model thus derived. ISAAC has the capacity to draw diagrams based on the model, and does so later in its execution. The major point, though, is that problem solution is critically dependent upon construction of an integrated, elaborated representation of the original problem statement. Since the way in which the problem is understood or represented so seriously impacts subsequent solution, it may be suggested that skilled solvers probably represent problems in ways that facilitate their solution--that is, the problem space generated itself suggests promising subsequent actions. In particular, how and what information is extracted from the problem statement, how it is organized and elaborated with auxiliary information, and how irrelevant information is identified and excluded, all would affect solution processing. In this sense, it may be that skilled problem solvers incorporate planning processes in the initial construction of the problem representation. The relationship between planning and problem representation is as yet unclear, however, and constitutes an interesting issue for investigation.

The initial qualitative analysis phase described by Larkin clearly does include a type of planning required for efficient technical problem solving. Execution of preliminary traces to identify the most promising solution procedure apparently is performed only by fairly sophisticated problem solvers. Less skilled individuals seem to plunge into the details of equation manipulation without first eliminating possibly unproductive solution paths. Sacerdoti's (Note 1) computer program, NOAH, embodies a set of mechanisms that accomplish this sort of planning. NOAH is a hierarchical planning system that develops and expands plans to increasingly detailed levels of specificity. Crucial to the emergence of an effective final plan is the functioning of "constructive critics" that examine the most developed plan currently under consideration. Each potential action in a plan has associated with it the preconditions, purposes, and consequences of that action. In the plan development process, the interaction of the planned actions is examined (criticized) in terms of their associated implications, and conflicts, redundancies, or inconsistencies are eliminated by reordering, addition of constraints, or elimination of redundant operations. Thus, a self-regulating, critical mechanism governs the construction of purposeful plans. In terms of technical problem solving, it may be said that the skilled problem solvers consider the requisite conditions along with the outcome or consequences of applying particular solution procedures in terms of the purpose or goal pursued (i.e., unknown variables for which quantities are sought). An on-going process of criticism is probably functioning to insure that the procedure(s) selected interact in a way that allows the problem goal to be reached. The development of this critical capacity is a crucial area for further investigation in understanding the nature of skilled problem solving.

Arithmetic word problem solving. Perhaps because of the difficulties involved in obtaining data from young children that allows for process analysis, little work has been conducted so far in this area. We have begun a project to study and model the process of solving arithmetic story problems, and can report some preliminary findings here. In this work we are collaborating with Mary Riley.

As described earlier in this chapter, a model of solving word problems was developed by Bobrow (1968); we have designed a computer simulation that models quite a different strategy of skilled word problem solution (Heller & Greeno, Note 5). Bobrow's system relied almost exclusively on syntactic information to translate problem texts into simultaneous algebraic equations for solution. Our system models semantic processing as the main component of problem understanding. The system abstracts the underlying quantitative structure, or schema, from problem passages, using a specialized method of reading comprehension. That is, the system constructs a semantic network representing the information in the problem in a way that explicitly facilitates problem solution. Arithmetic operations are selected on the basis of the structural representations constructed; there is no intervening process of building equations before the operation is selected.

The set of schemata that represent alternative structures of quantitative information were derived from analysis of addition and subtraction word problems found on standardized tests and in elementary school curricula. We have identified three distinct schema that appear to be necessary and sufficient for representing the structures of all one-step addition and subtraction problems we examined. These schemata can be referred to as Cause/Change, Combination, and Comparison.

Cause/Change problems are those containing situations in which some event changes the value of a quantity. For example, statements like "Jan had three lamb chops; Joe gave her four more," demonstrate a change in the quantity of objects possessed by one person as a result of an action. The abstract schema for this type of situation contains three main components: an initial quantitative state, an action involving a change value, either in the direction of increase or decrease, and a resulting quantitative state. The direction of the change, as well as the quantity that is unknown (initial, change, or result), govern the mathematical operation needed.

Combination situations are those in which there are two separate amounts that comprise a third, combined value. For example, in the problem, "Ray saw six birds. Jill saw four birds. How many birds did they see all together?" the two separate quantities must be combined to determine the value of the unknown total quantity. In Combine problems, the choice of an operation depends on whether the unknown value is one of the separate amounts or the combined amount.

The third schema involves two amounts that are compared and the difference between them. Examples of Comparison problems include, "Sue rode eight miles. Sam rode four miles. How many more miles did Sam ride than Sue?" or, "Rose ate nine cookies. Jim ate three fewer cookies than Rose. How many cookies did Jim eat?" The operation to be performed depends on the direction of the difference (more or fewer), and the quantity that is unknown (the difference or one of the separate quantities).

In our problem solving model, the problem text is translated first into a parsed form, in Anderson's ACT formalism (Anderson, 1976). As each problem proposition is read, it is incorporated into a gradually constructed abstract structure of one of the three types described above. This is accomplished by a translation process somewhat akin to Novak's (1976) canonical object frame construction. That is, objects and actions are considered in terms of their functions in the problem situation. Thus, the entire problem context is considered, but a very abstract representation of only the quantitative relations is constructed based on categorical information stored about verbs, types of phrases, parts of speech, etc., in the system. Details of particular names, verbs, and locations, for instance, are dropped as irrelevant to the schema. The crucial quantitative relations are inferred from the semantics of the problem while inferences that would be made in other contexts, such as ordinary story comprehension (e.g., Schank & Abelson, 1977) do not occur.

When a semantic representation has been constructed, the system infers the appropriate arithmetic solution procedure. It does this by referring to stored information about which operations are associated with which schemata, considering the direction of change or difference and the location of unknown quantities in the representation. At this point, our understanding of the details of this process are largely speculative and undergoing revision as empirical evidence is collected. We will therefore reserve a description of the process for later reports.

The most notable aspects of our theory derive more from its global solution method than the details of its operation. We are postulating that word problems can be solved without generating equations, and that construction of internal representations of the underlying quantitative relations to guide the solution are foremost in effective problem solving. These notions are supported by data showing that children can solve word problems before they begin to learn arithmetic and have any knowledge of equations (Buckingham & Maclatchy, 1930). Our own research (Riley & Greeno, Note 6) confirms this finding, with the addition of evidence that the semantic schemata involved in problems are rather strong determiners of problem difficulty for these children. In one experiment, second-grade children were found to have little difficulty with any problems with the Cause/Change structure or Combination problems with the combined amount unknown. Students were not as successful with Combination problems with one of the separate amounts unknown, and all of the problems having Comparison structures were relatively difficult for these young children. These results strongly support the notion that construction of accurate problem representations is of great importance in attainment of correct solutions. This process relates strongly to recent developments in the theory of natural language understanding (Schank, 1972; Winograd, 1972). "Understanding" is considered by these theorists as a constructive process: a situation is said to be understood when a complete and coherent structure representing the objects, concepts, and relations therein is constructed. Greeno (1977) has proposed that integration of these ideas with theories of procedural knowledge allow for more complete conceptualization of problem solving with understanding. In fact, much of the most recent work in problem solving described in this chapter has attempted to use this approach, and a general theory of skilled problem solving performance is thereby emerging. More specific distinguishing features of problem solving at different levels of expertise have also been discussed. In the following section we will attempt to synthesize these findings into a coherent picture of word problem solving processing.

Heller, Greeno

## Processing in Word Problem Solution with Understanding

Word problems can be grossly characterized as well-structured problems of transformation; definite initial and goal states are both given, and there exists a set of possible operators for moving toward solution. However, mediating representational and initial elaboration processes are required to transform word problem texts into problem spaces suitable for solution with understanding. A procedure for apprehending relations among components allows construction of integrated problem representations in the form of relational structures. Problem solving then requires search for the proper set and sequence of operators (arithmetic operations, algebraic manipulations, scientific principles and/or equations) leading to the final goal state. Equations are evoked or constructed and instantiated on the basis of these analytical processes.

It is becoming clear that the nature of problem solving processing differs as a function of both the expertise of the solver and the difficulty of the problem. That is, the nature of problem solving processes required cannot be determined in isolation from the particular problem solver engaged in working toward its solution; the problem solving strategies of an individual appear to vary depending on the complexity of the task, and different individuals working on identical problems differ in their behaviors. This view of problem solving can be extended to the domain of simple arithmetic; we have proposed that even basic arithmetic operations require complex problem solving processes, especially for beginning students. Although this is not a new concept, the specific forms of problem solving, given different combinations of solver skill and problem difficulty, are just beginning to be specified at a detailed level of analysis. With respect to word problem solution, the literature described in this chapter begins to suggest a general picture of these variations in problem solving behavior.

Problem solving skill may be viewed in terms of the solver's increasing capacity or competence for handling problems. Although numerous studies have undertaken to distinguish between "expert" and "novice" performance in various problem solving domains (e.g., deGroot, 1966 and Chase & Simon, 1973, in chess; Simon & Simon, in press, and Larkin, Note 4, in physics word problem solution), the findings of such analyses are best used for specifying a developmental continuum, i.e., a model of stages or levels in problem solving ability rather than only dichotomous distinctions between high and low skilled performance. The major dimensions along which these differential abilities vary in word problem solution are: the "understanding" processes, i.e., construction of problem representations; the organization of relevant knowledge, including problem category information or presence of chunks of scientific principles; the nature of initial analysis and planning processes; and the overall type of solution strategy used. A chart summarizing the solution features at different levels of competence is provided in Table 1.. Where evidence exists in the literature regarding the solutions described, a citation is provided in the rightmost column. The remaining descriptions are provided as intuitive propositions that remain to be tested empirically.

## Table 1

## Features of Word Problem Solution at Varying Levels of Competence

| PROBLEM TYPE | PROBLEM REPRESENTATION | KNOWLEDGE STRUCTURE | INITIAL ANALYSIS | SOLUTION STRATEGY | SOURCE |
|---|---|---|---|---|---|
| Non-technical | Rely on verbal problem statement (even if diagram constructed); no use of semantic and auxiliary information. | No problem category information used. (Can't tell if available.) | Read problem. | Direct, phrase-by-phrase translation from written problem to equation. | Paige & Simon, 1966<br>Hinsley, Hayes, & Simon, 1977 |
| Technical | | No evidence of schemata; principles available as discrete bits, no qualitative understanding. | No informal qualitative analysis; identify unknown and given values. Go directly to equations. | Nearly random search for equations containing desired quantities; "plug in values" solving for subgoals until answer is isolated. | Simon & Simon, in press |
| Non-technical | Some reliance on mediating representation and semantic information; reference to verbal problem statement as needed. | Some category information available; solution procedures not strongly associated with category information. | May recognize problem structure; search for solution procedure not strongly guided by category understanding. | Formally pursue several unproductive solution paths before problem solved; alternate between equation building and problem analysis. | |
| Technical | | Some principles organized into schemata; priorities not well defined. | Partial qualitative elaboration of problem features in relation to available schemata; select tentative procedure without efficient traces to eliminate ineffective plans. | Implement several plans for solution before problem solved; frequent iteration between analysis and equation working; inefficient solution. | Larkin, 1977 |
| Non-technical | Construct diagram when useful; develop integrated internal representation through qualitative analysis; rely only on mediating representation. | Strong problem category information available with associated solution procedures. | Recognize underlying problem structure; search for solution procedure associated with category. | Construct equations suggested by procedure associated with problem type. | |
| Technical | | Principles in well organized prioritized schemata with associated solution procedures. | Qualitative elaboration of problem features in relation to available schemata; search for matching schemata; preliminary traces to determine most promising solution procedure. | Implement plan for solution using selected schema; generate equations; rarely need to rework by alternative procedure. | Larkin, 1977 |
| Non-technical | Diagram often not needed; rely on strong, spontaneous mediating representation. | Strong problem category information available with associated solution procedures. | Problem structure immediately apparent; solution procedure evoked spontaneously. | Immediately generate and solve equations, solving for subgoals if necessary without hesitation. | Paige & Simon, 1966<br>Hinsely, Hayes, & Simon, 1977<br>Riley & Greeno, 1978 |
| Technical | | Principles in well organized prioritized schemata with associated solution procedures. | One appropriate schema immediately apparent on basis of qualitative problem features. | Immediately apply principles to generate and combine equations. | Bhaskar & Simon, 1977<br>Simon & Simon, in press. |

Solution characteristics are indicated separately for non-technical (arithmetic, algebra) and technical (scientific) word problems. This is to some degree an artificial distinction, since these problems tend to overlap depending on the extent of auxiliary information or principles required for solution. However, the distinctions are useful for examining behavior on clearly different classes of problems.

The continua along which solution features vary are abstracted in Figure 2. For the purposes of this general discussion, some of the details in Table 1 will be dropped and the more global variations emphasized.

The first feature, problem representation, has been discussed previously in terms of the notion of "understanding." Increasing levels of competence are characterized by greater reliance upon an integrated, qualitatively elaborated mediating representation of the problem structure. Syntactic processing in low level solutions, with heavy reliance upon the verbal problem formulation, contrasts with the semantic processing typical of individuals capable of problem solution with understanding. The integration and elaboration of structural problem representations with auxiliary and semantic knowledge is present in greater degrees at increasing levels of competence. Use of external or diagrammatic problem representations often characterizes higher skill solvers, but the critical factor is the reliance upon an internal understanding of the problem's underlying structure.

The organization of relevant knowledge or availability of schemata also becomes more salient with increasing competence. The strength of association between schemata and solution procedures increases as well, with prioritization of procedures more evident in higher ability problem solvers. That is, not only does the higher skill individual have available a broader range of schemata, but solution procedures are associated with the schemata and the procedures are ordered according to their usefulness with respect to different problem types and situations.

Initial problem analysis is a feature strongly related to problem representation, in that effective construction of representations reflects skill in and attention to preliminary elaboration of problems. In contrast to the low skilled individuals' rapid plunge into equation or number manipulation, higher skill individuals tend to consider a problem in an abstract problem space, planning a solution before working with the details. A sort of means-end analysis is sometimes performed, and potential solution procedures are examined before one is chosen as most promising for reaching the problem goal. This search for a solution procedure is highly organized with greater competence, as opposed to the low ability solvers' nearly random search for a combination of operators or principles that might yield the desired value. Lower ability solvers are not guided by an understanding of the problem and therefore cannot develop coherent solution plans with clear goal structures.

| | PROBLEM REPRESENTATION | KNOWLEDGE STRUCTURE | INITIAL ANALYSIS | SOLUTION STRATEGY |
|---|---|---|---|---|
| LOW SKILL | Rely on verbal problem statement; syntactic processing | No evidence of relevant schemata; solution information not associated with problem category knowledge | Read problem only | Piecemeal, direct translation; "plug in values"; search for solution combined with equation manipulation. |
| | ↕ | ↕ | ↕ | ↕ |
| HIGH SKILL | Rely entirely on integrated mediating representation; semantic processing with qualitative elaboration; problem "understood" | Strong available schemata; solution procedures associated with schemata and prioritized for application | Use information associated with schemata to identify solution procedures; most promising procedure either immediately apparent or determined by preliminary traces | Highly organized evokation of equations based on preliminary analysis and strong structural representation |

Figure 2 Continua of word problem solution variation.

It is interesting to note that the initial analysis phase is completed so rapidly in the highest level of competence that it was not explicitly noted in the literature. Although there was no overt evidence for this process in protocols of either very high skilled or low skilled solvers, we suggest that the surrounding behaviors imply considerably different explanations for the absences. In the case of lower skill solvers, the piecemeal, disorganized solutions suggest that no planning processes were executed. In the rapidly executed, high skilled solutions, however, the immediate generation of consistently correct and efficient solutions supports the notion that only one procedure was seen as applicable. In effect, the problem was so routine and simple that more than one procedure did not have to be considered--the solution was directly and singularly associated with the problem representation constructed. It cannot be concluded that the solution was obvious to the lower skilled individuals.

The overall solution strategy used varies in the degree to which planning and integrated structural representations determine equation construction. Lower skill solutions typically embody search processes in combination with equation manipulation; the solver is seeking a solution procedure at the level of instantiating equations; there is no reliance upon a clear understanding of the problem. In higher skill solutions, the search process is completed separately and before equations are constructed. Solution procedures are considered in relation to the problem representation, and no specific quantitative manipulation is attempted until a procedure is selected as promising. Therefore, unproductive solution paths would be pursued more often at lower levels of expertise, and solutions are generally less efficient than those carefully planned at higher ability levels. With respect to non-technical problems, direct translation using syntactic processing actually leads to incorrect solution of any problems requiring auxiliary information for accurate comprehension. In general, solvers relying upon the lower levels of solution strategies are limited both in the accuracy and efficiency of their problem solving repertoire.

The progression in problem solving ability sketched in these profiles needs both further empirical exploration and examination for educational implications. Once a developmental schema is firmly delineated, methods for facilitating transition to higher levels of competence may be developed and tested. One preliminary recommendation based on our current understanding of these stages would be to discourage any additional emphasis in the schools on direct translation processes. Because the highest competence levels rely strongly on a Gestalt-like understanding of the entire problem situation, and apprehending relations among problem components is hampered by piecemeal translation, procedures for constructing integrated problem representations need to be encouraged. Secondly, educators might best be sensitive to the solvers' need for well organized knowledge bases, containing concepts related by numerous explicit relations. Schemata comprised of related principles or problem category information appear to facilitate sophisticated problem solving efforts. Finally, it would appear that some means for imparting general procedural knowledge, including strategic approaches and problem solving heuristics, are called for. The highest skilled solvers are most facile at analyzing problem situations, qualitatively elaborating problems with auxiliary information, planning solutions in abstract problem spaces, and comparing features of problems with available schemata. These processes are typically not taught explicitly in educational settings; whether they should be, and if so, how they could be taught, are questions for empirical investigation.

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# Research on Children's Thinking and the Design of Mathematics Instruction

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The central assumption of this paper is that in order to systematically deal with fundamental problems of instruction in mathematics a clear understanding is needed of how children learn mathematics and what processes they use to solve mathematics problems. The argument that basic research on children's learning has an important contribution to make to the design of instruction has not always been universally accepted. As recently as the mid-sixties Gage (1964) could cite learning theorists of the stature of Estes and Hilgard to buttress his contention that learning theories have limited usefulness for education, and Suppes (1967) was forced to concede:

> Without question, we do not yet understand in any reasonable degree of scientific detail what goes on when a student learns a piece of mathematics, whether the mathematics in question be first-grade arithmetic, undergraduate calculus, or graduate school algebraic topology. (p. 1)

These pessimistic assessments reflect the fact that since the days of Thorndike most psychologists studying learning had turned their attention to simple learning situations that could be rigorously controlled. Tasks in most learning studies were selected for their experimental manipulability and ease of administration, which essentially excluded the types of complex learning situations inherent in most school curriculum. It is a small wonder that learning psychologists were conspicuously absent from the curriculum reforms of the late 1950's and early 1960's and scant attention was paid to their theories of learning.

In recent years, however, there has been a significant shift in emphasis in both the field of psychology and in the mathematics education community. Cognitive psychologists are increasingly turning their attention to the study of the complex tasks that comprise the school curriculum, and mathematics educators are also focusing to a much greater degree on how children learn mathematics and the processes they use to solve mathematics problems. There are currently at least four distinct lines of investigation dealing with children's acquisition of mathematical concepts, skills, and processes--Piagetian and neo-Piagetian research, research in information processing, the diagnostic-prescriptive movement, and large scale assessment projects like the National Assessment of Educational Progress.

There are fundamental differences between each of these four approaches in their basic assumptions about learners and education, in their research paradigms, in the type of knowledge about learning that they generate, and in their potential contributions to the mathematics curriculum. After briefly characterizing each of these approaches, this paper will discuss how this basic research on children's learning might be applied to the design of instruction in mathematics.

## Piagetian Research

By far the most extensive body of research on children's acquisition of mathematical concepts is based on the works of Piaget (see Carpenter, in press, for a more detailed discussion of Piagetian research and other cognitive developmental research). In the area of number, for example, Piaget's influence has been so great that it has led Flavell (1970) to observe, "Virtually everything of interest that we know about the early growth of number concepts grows out of Piaget's pioneer work in the area"(p.1001). Another measure of Piaget's impact is the sheer quantity of research based on his work. In the last three annual listings of research relative to mathematics education, almost a fifth of the reported research could be identified as Piagetian (Suydam & Weaver, 1976, 1977, 1978).

The great bulk of this research has been conducted by developmental psychologists, who are primarily concerned with the development of basic cognitive structures. Although these cognitive structures are often reflected in children's concepts of number, measurement, space, etc., it is not the purpose of most of this research to systematically describe children's acquisition of mathematical concepts. Most Piagetian research has attempted to account for or refute Piaget's initial findings. There has been relatively little effort aimed at expanding the range of mathematical concepts investigated or in identifying how the presence or absence of basic Piagetian constructs affect the learning of related mathematical concepts.

Most Piagetian research falls into one of three broad lines of investigation. Many of the early studies simply attempted to validate Piaget's observations with a variety of different materials and procedures. Other studies have attempted to identify and explain the sequence of development of related concepts, while a third broad class of studies has attempted to induce various Piagetian concepts via training. In general, Piaget's basic observations have been supported, and the behaviors that he describes are clearly more than experimental artifacts. However, most of the research indicates that there is a great deal less order in the development of related concepts than Piaget proposes. Finally, although many of the individual training studies have successfully induced various Piagetian concepts, they have failed to identify the specific mechanisms that do in fact lead to the development of the concepts.

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Piagetian research has successfully characterized children's performance on a variety of interesting problems involving fundamental mathematical relations. So far, however, this research has not established the link between the basic constructs identified by Piaget and the learning of the mathematical concepts and skills that comprise the core of the school mathematics curriculum. We know, for example, that at certain stages children fail to conserve or use transitive inference. However, we know very little about how the development of conservation and transitivity affects children's learning of basic number concepts and operations.

Most attempts to apply Piagetian research to educational problems have relied on inference. The analysis is essentially based on logical considerations. Since cardinal number is based on matching sets and the matching relation assumes conservation, conservation is a prerequisite for any meaningful concept of number. For Piaget it is a tautology that conservation is necessary for a meaningful concept of number. However, if one is concerned with children's ability to learn various number concepts and skills, the significance of conservation is not so clear and needs to be established empirically. The difficulty with a logical analysis of mathematical concepts is that children's logic is not the same as adult logic. If children are not asked specific conservation questions, they do not occur to them; and they ignore the fact that their judgments depend on certain prerequisite knowledge that they lack. There is ample evidence that children who are preoperational in Piagetian terms can successfully apply a variety of number, measurement, and geometric concepts and skills (Carpenter, in press).

For the most part developmental psychologists have focused on a very limited set of variables in characterizing the development of certain mathematical concepts, and they have not been interested in systematically describing how basic mathematical concepts develop through the course of instruction. Unfortunately much of the Piagetian research conducted by mathematics educators has been virtually indistinguishable from research based on purely psychological considerations. Many of the studies have simply examined relationships between Piagetian variables without establishing the significance of any of the concepts under consideration for the mathematics curriculum. Others have attempted to demonstrate that training on certain Piagetian concepts is possible without ever considering whether such training results in significant savings transfer for the learning of related mathematics concepts that are part of the school curriculum.

One approach to establish a relationship between basic Piagetian constructs and children's learning of mathematics has been to correlate performance on a test of Piagetian tasks with some measure of mathematics achievement (cf. Cathcart, 1971; Dimitrovsky & Almy, 1975; Kaufman & Kaufman, 1972; Steffe, 1970). These studies have found high positive correlations, even when IQ is held constant (Steffe, 1970). Furthermore, performance on Piagetian batteries administered in kindergarten appears to be an excellent predictor of mathematics achievement as much as two years later (Bearison, 1975; Dimitrovsky & Almy, 1975). These studies fail, however, to identify specific cause and effect relationships.

High positive correlations between performance on Piagetian tasks and arithmetic achievement do not imply that mastery of these tasks is a prerequisite for learning arithmetic concepts. In fact such a conclusion is completely inappropriate (cf. Carpenter, in press; Ginsburg, 1975, 1977a).

In order for variables like conservation to be useful in making decisions involving the teaching of mathematics, a much more explicit relationship between conservation ability and performance on specific mathematics tasks is needed. One of the reasons that conservation is a useful developmental construct is that by identifying a child's ability to conserve, it is possible to characterize the child's performance on a wide range of related tasks. A major question for mathematics educators is whether conservation or some other operation can serve a similar function in describing children's ability to learn basic mathematical concepts. There is ample evidence that conservation is highly correlated with arithmetic achievement. There is also ample evidence that nonconservers can learn many arithmetic concepts and skills. The basic research question is whether the ability to conserve is a prerequisite for learning certain basic concepts or skills or whether the high correlation between conservation and achievement is simply the result of a high correlation between conservation and general intelligence.

A study which illustrates this type of research is the teaching experiment recently reported by Steffe, Spikes, and Hirstein (Note 1), which identified specific differences between conservers and nonconservers in their ability to transfer various counting strategies to unfamiliar problems. At the Wisconsin Research and Development Center we are also attempting to identify whether certain Piagetian operations are a prerequisite for learning specific number concepts or skills. As part of our study we are attempting to characterize the processes used by operational and preoperational children to solve various addition and subtraction problems in order to uncover the relationships between certain Piagetian constructs and arithmetic concepts or skills. It is hypothesized that children's ability to conserve or apply transitive inference or class inclusion relations should be reflected in problems or solution strategies that make explicit demands on those constructs. Solutions involving simple counting stategies may not reflect these differences to as great a degree.

Another line of neo-Piagetian research, exemplified by the work of Gelman (1972a, 1972b, 1977) and Ginsburg (1977a, 1977b) has de-emphasized the importance of basic underlying concepts like conservation. They propose that the development of mathematical concepts can be more productively described in terms of the increasingly efficient application of specific skills like counting. This approach, which builds more on Piaget's clinical research techniques and general orientations to children's cognitive development than on explicit Piagetian theory, also holds real potential for studying concepts that are a central part of the mathematics curriculum. Thus, while early Piagetian research in mathematics education lacked direction in attempting to address problems of education, there is evidence of the emergence of a more focused attempt to investigate how the theories and techniques of Piaget might be applied to the study of education.

## Information Processing

There is a wide range of information processing theories. Although they are all based upon an analogy with the computer, some carry this analogy further than others. At the most task specific level, the goal is to actually construct a running computer program that models some segment of behavior. At the other end of the continuum, the computer serves as little more than a metaphor.

The prominent features of the general architecture of most information processsing systems include a short-term memory, which is extremely limited in capacity, and a long-term memory, which is potentially unlimited in capacity. The information processing system also has access to the external environment and some sort of mechanism for controlling attention that determines which sensory information is selected for processing. The long-term memory contains conditions or rules for processing information. All processing occurs serially in the short-term memory, and information from the external environment or long-term memory must enter the short-term memory before it can be acted upon.

There are two fundamental contributions that information processsing theory might make to our understanding of how children learn mathematics. One is to identify general limits of children's ability to process information. Since all information must pass through and be operated upon in the short-term memory, the capacity of children's short-term memory appears to be a potentially limiting factor. Pascual-Leone (1970, 1976) hypothesizes that the basic intellectual limitation of children is the number of bits of information they can handle simultaneously. The maximum number of discrete chunks of information that a child can integrate is assumed to grow linearly in an all-or-none manner as a function of age. From the early preoperational stage (3-4 years) a child's information processing capacity, or M-power, grows at the rate of one chunk every two years until the late formal operational stage (about 15-16 years). There are two problems involved in such an analysis. First, most children seldom operate at full capacity. Second, by chunking different bits of information together, children may essentially increase their processing capacity. As a consequence, the factors that affect children's ability to chunk information may be more critical than the capacity of their short-term memory. But whether it may ultimately be characterized in terms of processor capacity or chunking strategies, children's ability to integrate and simultaneously deal with different bits of information seems to be an important area for further study.

The other potentially productive information processing variables are children's ability to control attention and memory (Flavell, 1977; Hagen, Jongeward & Kail, 1975; Peck, Frankel, & Hess, 1975). Many of children's errors in traditional concept development tasks result from their attending to inappropriate dimensions of the problem. Individuals are faced with an overwhelming quantity of information from the environment which must be routed through the central processor in order to be acted upon. This potentially creates a tremendous bottleneck, and the mechanisms of attention which determine which information will be selected for processing are exceedingly important in characterizing information processing capacity. To plan instruction, it is essential to know the characteristics of the stimuli children can and naturally do attend to and how capable they are of shifting their attention from one dimension to another.

Memory is also an important information processing variable. As children mature they use increasingly efficient coding, storage, and retrieval strategies and are increasingly aware of the demands that specific tasks place on memory and their own ability to deal with them. Mathematical problems place significant demands on efficient use of memory, and inefficient use of memory may tend to clog the central processor, whose full capacity may be needed to solve the given problem. For example, it is quite difficult for most adults to multiply in base 8, even if they are given preliminary instruction in different number bases and are provided with a multiplication table. To some degree this simulates a very inefficient memory strategy.

The second potential contribution of information processing research is to explicitly characterize the processes that children use to solve various problems at different stages in the learning of a given concept. There is a fundamental difference between an information processing analysis of children's problem solving behavior and the approach that emanates out of Piaget's work. Instead of analyzing behavior in terms of the logical and algebraic properties of the problem, tasks are analyzed in terms of their information processing requirements. This distinction can be illustrated by an example from a study of children's measurement concepts (Carpenter, 1975). Since measurement of capacity involves transformations of the quantity being measured, it would be logical to conclude that conservation would be prerequisite for learning most basic measurement concepts; but this is not the case. Nonconservers can deal successfully with a variety of measurement problems. However, when equal quantities are measured with different units, a conflict situation is created that corresponds to the conflict created in the traditional conservation problem. Performance on this task closely parallels performance on the corresponding conservation problem because the two tasks have similar information processing requirements.

In general, information processing analysis attempts to describe learning in terms of specific concepts or skills which are placed into a system of hierarchically ordered subskills. Performance on individual problems is broken down into a series of discrete steps that can be serially ordered and often quantified. A number of techniques have been employed to identify children's problem solving processes. These include the analysis of problem solving protocols as well as the analysis of errors and response latencies. In simple information processing models, like the counting models for addition and subtraction (cf. Groen & Resnick, 1977; Suppes & Morningstar, 1972; Woods, Resnick, & Groen, 1975), processes are broken down into a number of identical steps, and the response latency for a given task is assumed to be a latency function of the number of steps needed to complete the task. Different models can be evaluated by regression equations generated from each model. More complex problem solving processes are frequently represented as production systems which can be translated into computer programs. If a running computer program can be constructed that effectively models human behavior, it provides an existence proof that the analysis underlying the program is at least a viable description of the processes used by human subjects.

An essential feature of either analysis is that behavior can be broken down into a set of discrete skills that are linearly ordered in the performance of any tasks. Such an analysis is based on an entirely different model and set of attendent assumptions than is Piagetian theory. Information processing theory is essentially mechanistic although it takes a very complex machine, the computer, as its analog. Piagetian theory, on the other hand, is more analogous to a biological organism. This organismic theory views the cognition in terms of integrated structural systems rather than as a series of discrete skills. According to this perspective the whole is greater than the sum of the parts, and it is not possible to understand these systems in terms of isolated skills. Which model is most appropriate is essentially a philosophical question rather than an empirical one. Reese and Overton (1970) propose that these models represent two independent world views that are based on different sets of assumptions and are essentially irreconcilable. However, research paradigms, the way in which behavior is characterized, and the implications one draws from the research for educational practice are all profoundly affected by the model one selects.

## Diagnostic-Prescriptive Mathematics

The diagnostic-prescriptive movement in mathematics education does not grow out of a research base, and until recently fundamental issues of the movement have not been defined in research terms. In the last several years there seems to be some shift towards developing a more substantial research base as is evidenced by the title of the newly formed Research Council on Diagnostic and Prescriptive Mathematics. But the movement is still strongly oriented to the immediate problems of the practitioner.

The basic tenet of diagnostic-prescriptive mathematics is that instruction should be based on an individual child's strengths and weaknesses in mathematics (cf. Ashlock, 1976; Reisman, 1972). Traditionally the focus has been on the diagnostic process; and little progress has been made in developing a taxonomy or categorization scheme for errors, identifying the prevalence of specific errors, or describing the relation between errors on different problems. Although diagnostic-prescriptive mathematics is loosely defined and it is inappropriate to generalize about the entire movement, a prevalent assumption seems to be that errors result from learning incorrect algorithms. Using a medical analogy, errors are seen as something to be diagnosed and remedied. This position contrasts markedly with that of Piaget, who in some sense recognizes the legitimacy of children's errors. Piaget does not regard the responses he observes as errors to be diagnosed and corrected. Rather they are reflections of children's basic level of cognitive functioning. Romberg (Note 2) has argued that this focus on the superficial aspects of children's errors is too narrow and a second level of diagnosis is needed that starts to take into account children's general level of cognitive functioning and the deeper structure of their solutions.

Diagnostic-prescriptive mathematics clearly should not be an independent line of investigation distinct from either Piagetian or information processing research. Some sort of interface could mutually enhance all three approaches to the study of children's learning of mathematics. Piagetian and information processing research could provide a broader and more rigorous basis for diagnosing children's mathematical abilities, while diagnostic-prescriptive mathematics provides one mechanism for applying basic research on children's learning to instructional practice. Perhaps the greatest contribution of the diagnostic-prescriptive movement has been to reorient practitioners to see the problems of instruction in terms of the learner.

## Large Scale Assessments

One of the limitations of most of the research on children's mathematical thinking is that individual researchers generally lack the resources to representatively sample from the general population. In fact, special samples of children who clearly articulate their problem solving processes are often selected for clinical studies. Although they have their own limitations, large-scale studies help to round out the picture of children's mathematical thinking by describing how prevalent certain errors or processes are in the general population. There have been a number of large-scale assessments of children's mathematical abilities including The International Study, The National Longitudinal Study of Mathematical Abilities (NLSMA), and various state assessments. The mathematics assessment of The National Assessment of Educational Progress (NAEP) probably provides the most representative sample of children's mathematical performance of any of the large-scale assessments (Carpenter, Coburn, Reys, & Wilson, 1975a, 1975b, 1978).

National Assessment does not build upon any theory of learning or development and generally does not attempt to test hypotheses about children's learning or the underlying causes of their errors. Instead it attempts to provide base-line data on the performance of 9-year-olds, 13-year-olds, 17-year-olds, and young adults on a variety of individual exercises over a range of mathematical topics. Many of the exercises are open-ended, and analysis includes identifying the frequency of specific errors as well as the frequency of correct responses. One potentially significant finding of the first mathematics assessment was that many fewer systematic errors were found than the results of most clinical studies and the literature of the diagnostic-prescriptive movement would suggest. Whether the standardized test administration and response analysis techniques mitigate against finding such errors or whether systematic errors are significantly over-represented in most clinical studies is an open question that warrants further investigation.

Since the mathematics assessment is scheduled on a five-year cycle, National Assessment should provide a reasonably up-to-date measure of children's performance. The second mathematics assessment has just been completed. Although it does not assess performance of young adults and does not include individually administered exercises as in the first mathematics assessment, it provides a much more comprehensive sampling of major content domains. Because of the extreme care taken to select a truly representative sample, the results of National Assessment are a unique resource. Although these results do not provide the detailed insights into children's mathematical thinking that may be generated from other studies, researchers need to take the results from National Assessment into account in order to establish the generalizability of their own findings. There is a need for additional research specifically directed at explaining and extending the results from National Assessment. Such a program has been discussed in some detail in another article (Carpenter, Coburn, Reys, & Wilson, 1976).

## Potential Educational Applications

The two most promising areas of application of research on children's mathematical thinking involve the selection and sequencing of content and the individualization of instruction.

### The Selection and Sequencing of Content

There are several ways that research on children's learning of mathematics might be applied to the design of instruction. One of the most obvious is to somehow incorporate into the curriculum the solution processes that children spontaneously generate for themselves. For example, independently of instruction young children develop quite sophisticated counting strategies for solving simple addition and subtraction problems (cf. Groen & Resnick, 1977; Suppes & Morningstar, 1972; Woods, Resnick, & Groen, 1975). Traditionally, instruction has failed to give serious consideration to the richness and growing sophistication of these strategies. However, curriculum could be developed that builds upon these strategies rather than portraying arithmetic operations exclusively in terms of set operations. In fact, Steffe et al. (Note 1) have already done so.

There are a number of ways that curriculum may be designed to build upon children's spontaneous strategies. One is to identify a minimum set of concepts and skills that all children exhibit at one point or another in the acquisition of a given topic and build instruction around this basic set. This approach is not especially elegant and seems to reduce instruction to the least common denominator. However, one might assume that if one teaches the minimal set of skills that is logically complete and which can be understood by all students, the better students will continue to generate their own more complex strategies. The study by Groen and Resnick (1977) offers some support for this hypothesis.

An alternative approach would be to identify the most efficient processes that children use and/or the processes that are used by the most capable students and specifically teach these processes. Although this approach has the appeal of attempting to make the most efficient strategies available to all the students, there are some potential drawbacks. The slower students may not have the cognitive capacity to understand or apply the complex processes of the better students. Additionally the complex processes may be very difficult to teach explicitly.

Clearly these extremes do not represent the only choices, and there is a great deal of middle ground. Furthermore, as Resnick (1976) proposes, appropriate instruction should not necessarily copy the spontaneous development of the concepts in children. Instead, it should put learners in the best position to invent or discover appropriate strategies themselves.

There is clearly no simple answer to the question of how to build upon children's spontaneous strategies, and it is unlikely that a single approach will be effective with all content or for all learners.

A second potential application of knowledge about the natural development of mathematical concepts involves the selection of a general approach or model and decisions involving the relative sequencing of major topics. In this case decisions do not simply involve whether to include instruction on specific strategies but whether certain general models are more intuitive than others or whether certain topics naturally develop before others. The decisions are much more global and reflect a more holistic conception of cognition characteristic of organismic models.

An example of a line of investigation that illustrates the application of this paradigm is the work of Brainerd (1973a, 1973b, 1976). He proposes that basic natural number concepts can be developed logically from an ordinal perspective as evidenced by the work of Dedekind and Peano or from a cardinal perspective in the tradition of Russell and Whitehead. Based on his research with young children, Brainerd contends that ordinal number develops before cardinal number, and ordinal number is more closely connected with the initial emergence of arithmetic. Based upon these conclusions, Brainerd recommends that serious consideration be given to abandoning the traditional cardinal development of natural number in favor of ordinal definitions. Even if Brainerd's conclusions regarding the emergence of ordinal and cardinal concepts were valid, his conclusions would represent unwarranted extrapolation. No attempt was made to actually design and test instruction based onthe ordinal definition of number. Furthermore, the examples of ordinal and cardinal concepts included in Brainerd's studies represent only a very narrow sampling of the concepts involved in the development of either ordinal or cardinal numbers.

On the whole decisions involving radical restructuring of major topics seem to require a rather high level of inference and also involve value judgments regarding the relative importance of certain topics or ways of developing given concepts. Consequently this approach seems to hold less potential than the program outlined previously.

An alternative to focusing on children's naturally developed concepts and successful strategies it to analyze their errors. By identifying serious misconceptions or significant prerequisite concepts or skills that children are failing to master, instruction may be designed to compensate for these difficiencies. The series of studies by Gal'perin and Georgiev (1969) is an excellent example of research of this type. In an initial study they identified many of the same type of conservation and measurement errors found by Piaget. But rather than accepting these errors as developmental phenomena, they attributed them to the traditional emphasis in school mathematics programs on number concepts, which incorrectly characterized units as discrete entities.

To test their hypothesis, they administered a series of measurement problems to the "upper group" of a Soviet kindergarten. They concluded that young children who are taught by traditional methods lack a basic understanding of a unit of measure. They do not recognize that each unit may not be directly identifiable as an entity and that the unit itself may consist of parts. They are indifferent to the size and fullness of a unit of measure and have more faith in direct visual comparison of quantities than in measurement by a given unit.

On the basis of this study, Gal'perin and Georgiev devised a program of 68 lessons that focused on measurement concepts and systematically differentiated between units of measure and separate entities. The lessons were divided into three parts. The first part dealt with forming a mathematical approach to the study of quantities. This section focused on replacing the habit of direct visual comparison with systematic application of measuring units. Appropriate units for measuring different quantities were identified and measuring skills were studied directly, with special attention directed to the deficiencies identified in the pretest. A variety of units was used, including units consisting of several parts (two or three matches, spoons, etc.) or some fractions of a larger object (half a mug or stick). All of these concepts were presented without assigning numbers to the quantities.

It was not until the second part that the concept of number was introduced. Thus, Gal'perin and Georgiev introduced most of the basic measuring skills and spatial concepts before they introduced numbers. In the third part, the inverse relationship between the size of the unit and the number of units was introduced.

Although the investigation was not conducted with strict experimental controls, the students who participated in this program showed striking gains over the performance of the previous year's students. Whereas fewer than half the students in the previous year could answer most of the items on the measurement test, performance was close to 100 percent for the experimental group.

A fourth perspective is that, while research on children's thinking may have profound implications for educational practice, it is inappropriate to attempt to apply such research to design specific instructional programs.

> You make a great, a very great mistake, if you think that psychology, being the science of the mind's laws, is something from which you can deduce definite programmes and schemes and methods of instruction for immediate schoolroom use. Psychology is a science, and teaching is an art; and sciences never generate arts directly out of themselves. An intermediary inventive mind must make the application, by using its originality. (James, 1939, p. 7-8)

From this perspective research on children's mathematical thinking is important, for instruction must be consistent with the ways in which children learn. But there is no single pattern of instruction which is most appropriate. Children's learning, the instructional process, or both are considered to be too complex for any a priori specification of curriculum. The problem therefore becomes one of providing teachers with sufficient knowledge of children's thinking so that they can evaluate the specific needs of their own students and implement instruction accordingly. This essentially is the approach typical of much of the diagnostic-prescriptive movement, with its emphasis on diagnostic skills.

## Individualization of Instruction

A second potential contribution of research on children's mathematical thinking is to provide a basis for individualizing instruction. Whereas decisions involving the selection and sequencing of content need to be based on common elements of at least some children's thinking, individualization would be based on identifying individual differences between children. In the past, global measures like IQ have been used to separate children for instruction because these measures correlate highly with school learning. Individualization needs to be based on a much more careful analysis of how specific abilities limit children's capacity to learn specific mathematics content in a specific way.

Three basic steps are required. First, it is necessary to construct a good measure of children's mathematical ability. Next specific mathematics tasks must be analyzed in terms of their demands on the ability. Finally, instruction must be designed that provides tasks that are appropriate for the different levels of the given ability.

A variety of alternatives for developing measures of children's thinking are possible. The simplest and most straightforward involve identifying children's knowledge of prerequisite skills. A somewhat more sophisticated approach includes the analysis of the specific processes that children apply to solve mathematics problems. This is essentially the approach of diagnostic-prescriptive mathematics at its best. Since the processes and errors are a function of the specific mathematics topic being learned, frequent assessment of individual children is required. Consequently, the development of efficient procedures for identifying children's processes and errors is a critical problem.

A second alternative is to base such a measure on fundamental development variables like conservation, class inclusion, and transitivity that are presumed to develop outside of formal instruction. Several of these measures have been shown to correlate highly with mathematical achievement (Carpenter, in press). However, with the exception of the study by Steffe et al. (Note 1), little progress has been made in relating these measures to children's ability to learn specific mathematical concepts and skills. In other words, it is not sufficient to demonstrate that there is a difference in overall achievement between conservers and nonconservers. It is necessary to document exactly how they differ and what instruction is appropriate for each group.

A third potential approach would involve some measure or combination of measures of information processing capacity. Case (1975), for example, has described how instruction might be redesigned to fit different levels of information processing capacity. He proposes that instruction can be redesigned so that the maximum number of subskills that the learner has to coordinate at any given time is minimized.

Another potential measure is Vygotsky's (1962, 1978) zone of proximal development. This is defined by Vygotsky (1962) as "The discrepancy between a child's actual mental age and the level he reaches in solving problems with assistance "(p. 103). Since this measure actually involves adult interaction which represents a form of instruction, it should potentially provide an excellent measure of children's ability to benefit from instruction.

A closely related technique is the applicaton of teach-test procedures to ascertain children's ability to deal with certain types of instruction. Teach-test procedures have frequently been used with mentally retarded children to measure their susceptability to traditional forms of instruction (cf. Budoff, 1967) but have seldom been used with normal children. The basic format involves a short, controlled training session over certain novel and presumably unfamiliar tasks followed by a test on the instructed material. Unlike other measures the initial knowledge or ability to do the task is not the primary concern. What is of interest is to what degree subjects are able to profit from the instructional sequence. By manipulating the form of the short training session, one potentially can generate a measure of children's ability to attend to and learn from different instructional sequences.

A study that illustrates the application of this technique is reported by Montgomery (1973). This study was an aptitude-treatment interaction study which examined the interaction of second- and third-grader's ability to learn unit of length concepts with two treatments based on area and unit of area concepts. Aptitude was measured using a teach-test procedure which partitioned subjects on their ability to learn to compare two lengths measured with different units. Subjects were randomly assigned to one of two nine-day instructional treatments on measuring and comparing areas. The difference between the treatments was the emphasis placed on the unit of measure. In one treatment, subjects always measured with congruent units and compared regions covered with congruent units. In the other treatment, subjects measured with noncongruent units and compared regions covered with different units. On both a posttest and a retention test, the treatment that used different units was significantly more successful in teaching children to assign a number to a region (measure) and to compare two regions using their measures. However, there was no significant difference between the two treatments on a transfer test that included problems involving measurement with different units, and no significant interactions were found between aptitude levels and treatments. The failure to find significant results may in part reflect certain anomalies in the development of measurement concepts that were not taken into account (Carpenter, 1976). But it does illustrate the difficulties and pitfalls in attempting to construct good measures in order to match children to appropriate instruction.

## Conclusions

Two general applications of research on children's thinking that seem to hold the greatest promise for influencing educational practice have been identified. The first involves the selection and sequencing of content. The second concerns the individualization of instruction. Both of these applications require a comprehensive cognitive map of children's learning of key mathematical concepts and processes. This map must take into account both individual differences and the effects of instruction. Thus, a major objective for research in mathematics education should be to characterize the processes and concepts that children acquire at significant points in the learning of important mathematical topics. Furthermore, it should describe how these concepts and processes evolve over the course of instruction. This involves describing the different processes and errors that individual children exhibit on key tasks at each stage of instruction. It also should include an analysis of performance on related tasks. Although significant individual differences should be anticipated, hopefully it will be possible to identify clusters of children who exhibit similar profiles of performance over a range of tasks. If this is the case, then key problems may be used to identify how individual children will perform over the complete range of tasks. Such problems may provide a basis for individualizing instruction.

Finally it is necessary to describe the change in concepts, processes, and errors over the course of instruction. Piaget assumes that all children go through essentially the same stages of development. Therefore it is only necessary to characterize each stage to describe development. The evidence suggests, however, that there is a great deal of variation in the pattern of acquisition of most mathematical concepts. Consequently, to characterize development of these concepts it is necessary to describe how change takes place within individual children or at least groups of children over the course of instruction. This means analyzing how certian processes, concepts, or errors at a given stage have evolved from the processes, concepts, or misconceptions of earlier stages. For example, if a child makes certain errors early in the learning of a concept, will they be resolved as the child acquires more mature concepts and skills, or will these errors be magnified as new concepts are built upon these earlier misconceptions?

To effectively assess change within individual children it is necessary to follow individual children over the relevant instructional periods. This does not mean that the only studies that are appropriate are longitudinal studies over the entire course of development of a given concept. But any study that purports to measure intra-individual change must at least have repeated measures on the same subjects over the time that change is being measured.

Individual children master concepts at different points in an instructional sequence. An important question is whether all children go through essentially the same basic sequence in learning certain concepts even though they may pass through a given stage at different points in the instructional sequence. In other words are there certain key prerequisite concepts or processes that all children achieve before they master a given concept. Research should be especially sensitive to identifying such key prerequisites.

Recent research has provided critical insights into children's understanding of certain key mathematics concepts and the processes they use to solve a variety of mathematics problems. But much of the basic content of the mathematics curriculum has not been systematically studied. Most basic psychological research has focused on content that fits certain psychological theories or can be neatly modeled. There is still a very incomplete picture of children's performance at critical stages in the acquisition of fundamental mathematical concepts and skills. In addition, there has only been limited success in characterizing children's performance on related tasks. Little progress has been made in identifying the specific factors that lead to the acquisition of more advanced concepts and skills or in accounting for individual differences.

The state of the art is such that many of these more complex interactions cannot be specified with the precision that is possible for more straightforward individual tasks. However, it is this kind of information that is generally most critical for designing instruction. This points to a significant difference between research in mathematics education and research based exclusively on psychological considerations. Psychological research can afford to focus on a narrow range of behaviors that can be carefully controlled and specified with precision. Research in mathematics education, on the other hand, needs to select its content based on considerations of the significance of the content within the mathematics curriculum rather than on experimental or theoretical grounds. Moreover, research in mathematics education should not be directed at incrementally extending existing theories of cognition or development. Research in mathematics education needs to begin to apply the principles of research in cognition and development to problems that are significant for designing instruction, even if such an application results in some loss of precision. Such an effort should be aimed at the construction of what Shulman (1974) has called middle-range theories. Such theories fall between the task specific working hypotheses that are generated to explain individual behaviors, errors, etc., and comprehensive theories such as those of Piaget that attempt to encompass all of learning or cognitive development.

Furthermore, the connection between research on children's mathematical thinking and the design of instruction cannot be made by inference alone. What is needed is what Glaser (1976a, 1976b) calls a "linking science" to establish this relationship. In other words fundamental instructional issues cannot be resolved directly on the basis of pure cognitive development research. Research on cognition and development is descriptive, not prescriptive. It does provide a basis for initiating certain lines of instructional research. But linking research cannot wait until a complete description of the development of a given concept is available. The process is iterative. As instruction based on research on children's mathematical thinking is designed and evaluated, further insights into children's mathematical thinking will be found; and new instruction will be designed based on these findings.

Finally, we must recognize that even if clear implications of research on children's thinking can be established, this does not guarantee that this research will have any impact on educational practice. Although we must avoid premature conclusions and clearly establish the links between research on children's thinking and classroom practice, we must not bury our results in research journals. Unless we can convince teachers and curriculum developers to begin to see some of the problems of education in terms of the learner, research on children's thinking will have little practical value for the teaching of mathematics.

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# A Trend in Problem Solving Research

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Much research and other formal documentation exists to show how mathematics instruction fails to develop problem-solving skills in children. The nature of the failure can, however, be illustrated by the less formal means of describing an incident that happened some years ago. The incident involved Bruce, then a ninth-grade student who lived next door. Bruce lacked general mathematical ability and did poorly with abstract ideas and problems in mathematics class. However, what he lacked in general ability he more than made up for in enthusiasm and eagerness to tackle problems of a more practical nature. Whenever any construction of any kind was going on in my yard, Bruce was there in short order.

On this particular day I was completing a workbench and Bruce was my willing helper. I had put aside a good piece of 1-by-4 lumber which I intended to cut into three strips of equal width to trim along the front of the bench. The bench was 22 feet long and the 8-foot board would provide just enough for the trim and the fitting. When Bruce understood what was to be done, he offered to mark the board for sawing.

His first step was to divide 4 (the width of the board in inches, so he thought) by three. He got 4/3, then a 1 1/3. The trouble was that the units in his calculations did not jibe with any of the units on the square he was using. He finally decided to estimate 1 1/3 inches and did indeed measure quite precisely. Nonetheless, the last mark was obviously much closer to the edge of the board than it should be. Anyone familiar with lumber knows that the width of boards is usually the width before planing. Milling of a 4-inch board reduces its width to about 3 5/8 inches. That seemed to explain to Bruce's satisfaction why the second mark was always closer to the edge than it should be.

When I suggested he measure the board and found it to be only 3 5/8 inches wide he looked a bit confused but went on with a revised calculation. He divided 3 5/8 (the measure in inches he had obtained) by 3 and though the computation gave him some trouble he finally got 1 5/24. Twenty-fourths were not marked on the square, of course, and he didn't even attempt to estimate. He just gave up.

Carpenters have a neat way of solving a problem like this without using arithmetic or written symbols. To mark the board into three pieces, they would take a whole number greater than the width of the board to be divided but also a number divisible by 3. In the case of a 4-inch board 6 would be a good choice. The square is then laid on the board obliquely as shown in Figure 1.

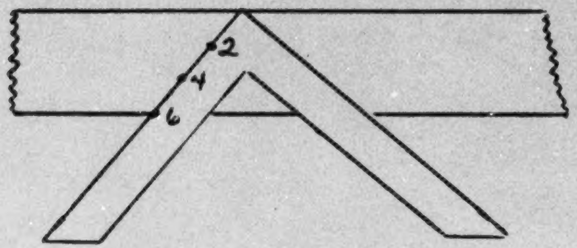

Figure 1

Note that the positioning of the square puts the vertex of the right angle on one edge of the board and the 6 on the other. Then marks are made at 2 and 4. The square is moved along the board and the process repeated. The marks are joined as shown in Figure 2.

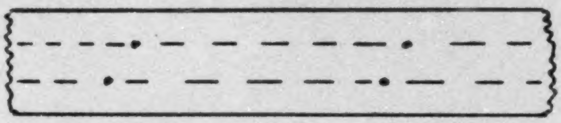

Figure 2

Sawing along the lines will provide three strips of equal width (providing marking and sawing is done with care--and assuming the width of the saw blade cut is a neglible error).

The underlying elementary geometric principle is usually taught in connection with the study of similar triangles but is sometimes introduced earlier in connection with parallel lines and transversals. Bruce had recently made a rather thorough study of similar triangles in his ninth-grade mathematics course so I preferred to relate it to that. The more I tried, the more obvious it became that Bruce was not buying that similar triangle thing. His question was: How could similar triangles be involved when you didn't even draw a triangle? After a few more half-hearted attempts it was my turn to give up, which I did.

The point to be made in describing this incident is to show how little apparent connection there is between mathematics instruction and problems which occur in practical everyday settings. Students faced with a practical problem of the kind described typically proceed blindly to compute and when that fails have no further problem-solving resources to use.

Why is mathematical instruction so deficient in developing problem-solving skills? What procedures would have to be developed to make the parallel structures of physical situations and related mathematical notions more apparent to the learner? What happens on the interface of real world experience and abstract mathematical structures which are the fabric of mathematics instruction? What research procedures and methods show the best potential for answering these and related questions? Finally, can answers to these questions be translated into instructional procedures which will substantially improve the problem-solving skills of children learning mathematics?

Not much can be said at this time about the last question. However, the object of this paper is to suggest a general procedure for doing problem-solving research and to describe some studies in which an attempt is made to apply the procedure. Something must first be said, though, about problem solving as it relates to the goals of mathematics instruction and also about the difficulties encountered by teachers who use practical problems as a vehicle for mathematics learning. The discussion will be mainly concerned with elementary school age children and younger.

## Problem Solving and the Goals of Mathematics Instruction

If we look in at any class of elementary school children there would be general agreement that the basic reason for giving them instruction in mathematics would be to help them solve problems which they are likely to meet in their daily living. Yet, the methods usually adopted in teaching mathematics at this level tends to foster the growth of a skein of mathematical ideas, process, and skills which seem to have little or no connection with the real-world experiences or problems faced by the child. Analysis of the results of the National Assessment of Educational Progress in mathematics (Carpenter, Coburn, Reys, & Wilson, 1975) revealed that while elementary school children had developed considerable computational skill, they lacked even such fundamental problem-solving processes as "checking the correctness or reasonableness of a result, or making an estimate. . .".

When we consider the development of problem-solving processes in children, it is well to remember that in the very early stages of mathematics learning there is general recognition that all basic mathematical ideas have their source in real-world experiences--so methods of teaching rely to a greater or lesser extent on real problems or at least on manipulation of concrete materials. However, after the first year or two of elementary school mathematics, teaching tends less and less to be concerned with real-world problems and their solutions, and more and more with computation and with symbolic or abstract aspects of mathematics. It can easily be demonstrated that these abstract and symbolic forms are largely inappropriate for elementary school children.

Let me hasten to point out this is certainly not an indictment of teachers presently offering instruction in elementary school mathematics. In the first place, research has very little to say to teachers about the precise role problem applications have in mathematics learning or how mathematical structures, once attained by the child, find easy application in the solution of everyday problems. In the second place, some so-called experts in early mathematics learning recommend that the processes and skills of mathematics should be learned and that applications can and should be found later. Their argument hinges around the real or imagined difficulty in finding any but trivial applications in the early stages. Finally, collections of applications and application ideas available to teachers are apt to include very few which would be appropriate at the elementary school level.

## Some Consequences of Basing Mathematics Instruction on Real Problems

In the face of the difficulties outlined in the previous section, let us take the case of a hypothetical teacher who decides to provide a problem-solving base of a practical nature in teaching mathematical processes. As an example, let us assume that the lesson is an attempt to construct a symbolic form for simple division using real problems. To keep it simple, the teacher decides to be concerned only with measurment division; that is, the form of division which specifies the number in each group and requires that the number of groups be found.

The teacher carefully constructs a layout which, let us say, consists of seven joined enclosures to represent stalls and fifteen horses which are to be placed in the stalls, three per stall. A protocol is then carefully worked out which specifies exactly how the problem is to be presented. This protocol, in short, tells the child how many horses are to occupy each stall and also asks the question, "In how many stalls will there be three horses?" On the surface this appears to be straightforward enough, but let us see what kind of difficulties may arise.

Most children would have little difficulty placing the horses as required in the stalls, counting the occupied stalls and supplying the answer. One should note that the manipulation can be done as easily whether the child knows the total number of horses to begin with or not. In fact, there is no reason apparent to the child for having the information. Yet, if this is to be related to the symbolic or computational form, most teachers I know (and I have no alternate recommendation to give them) would write down the number fifteen and proceed as follows:

Then

$\overline{)\,15}$ to indicate the division process.

Then $3\overline{)\,15}$ to indicate the number in each stall.

Then

Then $\begin{array}{r}5\\3\overline{)\,15}\end{array}$ to indicate that there will be horses in five of the stalls.

How can the process expressed in symbols be related to the actual manipulation or problem solution when the very first number to be written down in symbols is not needed at all to do the real problem? No research

I know provides any definitive help to the teachers in this situation, that is, the process of making a transition from concrete problem-solving procedures to symbolic computation procedures. Clearly research is needed to provide help in this direction.

The problem situation can be readily altered, but this can create new difficulties. To illustrate, suppose the teacher had decided instead of horses and stalls to set up a problem in which items of cargo had to be loaded onto a truck. The items might be fifteen blocks and the problem to load a toy truck from a loader which carries three blocks at a time and to find how many loads the loader will have to take to get all the blocks onto the truck. Changing the substance of the situation has not made the need to know the total number of blocks any more apparent. The real problem can be solved as well whether the total number to begin with is known or not.

But this particular situation has embedded in it a further peculiarity as well. There is no guarantee once the truck has been loaded that it would be apparent how many loads the loader had taken. The child would have had to know in advance that some means would have to be found which would preserve the integrity of each group so they could be counted in the end or some mental record would have to be kept of the number of loads. There is no such requirement in the horse and stall example.

No doubt children would as likely encounter one of these situations in reality as the other. But no research exists to indicate which might be more efficient in helping the child understand the process of division or seeing sense in its symbolic representation. Which situations are closer tied to computation? How we can get easy generalization to other situations or which examples should come first? We do not have information to provide the answers to questions such as these. One might even hypothesize that there is enough difference in the physical and symbolic solutions that learning the symbolic form might actually interfere with solving the problem with objects and vice versa.

Learning the process of division is further complicated because sometimes remainders are involved and the child has to exercise keen judgment in order to give a sensible result. Then there is the partitive form of division whose symbolic representation may be identical to that for measurement division but in which the requirement is to find the number in each equal group when the number of groups is known. For example, how many horses would be in each of five stalls if there are fifteen horses in all? Here again, the real-world problems could be solved readily by manipulation without ever knowing the total number of objects; while the symbolic form requires it as a start. Research has revealed that few children up to the age of eight or nine have an entirely systematic way of partitioning objects (Bourgeois & Nelson, 1977). Some start with a few objects and keep adding on until all the objects are used up. Others seem to try to make a good estimate--and sometimes are right without becoming involved in a one-by-one partitioning at all.

Others use a random trial-and-error process and some have no means of solving the problem. There is no research that directs the teacher in the best way to teach partitioning so it becomes a natural, systematic process for the child, readily related to a symbolic representation.

These examples serve to illustrate some of the complications which must be faced when a decision is made to use practical problems and their solutions as the basis for even a fundamental mathematical operation. If those outlined were the only complications encountered, the course of research could be fairly specifically mapped out. Some other complications are to be discussed later but are outlined by Bourgeois and Nelson (1977).

## Needed Foci in Research

In spite of the difficulties associated with using real problems to help build up mathematical ideas, few would recommend trying to teach mathematics without the use of such problems. In fact, if there had to be a choice, the child might better be served by learning the process in the real situation than to have to learn some vaguely understood symbolic form. Future research may reveal, in fact, that children need to have a very clear idea of how to solve various problems in real situations with real objects over an extended period of time before any attempt at all is made to render such solutions in symbolic form. In any case, children's misunderstanding of the meaning of symbols when introduced without a firm basis in real experience is well known.

A basic aim of elementary school instruction in mathematics is to assist children in solving various practical problems that occur in their daily lives; and such experiences, at least in the early stages, must provide the actual basis for mathematical learning and understanding. If we are to understand the complex interactions which occur on the interface of developing mathematical structures and related experiences in the real world, then it is clear that a great deal of emphasis must be placed on problem-solving research involving real and significant problems. Exhortation, testimonial and speculation must be replaced by empirical data which will provide more definitive guidance for planning learning experiences in elementary school mathematics.

## Developing a Research Methodology

The research methodology considered to have the best potential for answering the questions posed about the role of real problems in mathematcis learning is based upon a class of models which Easley (1977) describes as dynamic structural models. The particular model and related methodology discussed here has the following characteristics: It is clinical in nature and may or may not involve an interview. It involves a carefully constructed problem situation in which a single child is placed. Responses may or may not be verbal, though ones emanating from interviews would probably be largely verbal. In any case, while in the situation, all responses of the child are recorded on video tape (or audio tape if only the interviews are to be analyzed). Analysis is carried out by trained observers whose observations are checked by one or two other trained observers. Reports of analyses are largely descriptive in nature.

The carefully constructed problem situations are the key to this methodology and provide the framework within which data are to be gathered. An initial difficulty in designing such problems is that there is considerable difference of opinion about what constitutes a problem. Simply calling it a practical problem won't do much to clarify the meaning or resolve the differences. One effective way to overcome the difficulty is to develop a set of criteria or guidelines upon which the creation of the problem situations will be based. Liedke and Vance (1978) list these characteristics of a "good" problem:

> It is open ended. The problem can be interpreted in different ways, several procedures for arriving at the solution are possible, or more than one solution may exist.
> It provides for maximum involvement on the part of the pupils and minimum teacher direction.
> It leads to further problems.
> It can be interpreted into other areas of study. (p. 35)

Although their list is designed primarily for helping prospective elementary school teachers create effective problem situations for children, it appears that their criteria could be adapted readily for research purposes. At the University of Alberta the design of "good" problems used in problem-solving research has been based upon a set of seven criteria developed by Nelson and Kirkpatrick (1975). The criteria are as follows:

1. The problem should have demonstrable significance mathematically.
2. The situation in which the problem occurs should involve real objects or simulations of real objects.
3. The problem should be interesting to the child.
4. The problem should require the child to make some modifications or transformations in the materials used.
5. The problem should allow for different levels of solution.
6. Many physical embodiments should be possible for the same problem.
7. The child should believe the problem can be solved and should indicate when a solution has been reached.

Sets of criteria for problems and problem solving would probably differ from investigator to investigator and would no doubt reflect more precisely the individual's own concept of what a problem is. Provided criteria or guidelines are clearly stated and faithfully applied, the effects of any particular criterion could readily be tested as part of the analyses of the responses children make to the problems. Such evaluation would at least provide the means of constant refinement of the criteria. The important point to be made here is that differences in criteria or guidelines used by different investigators are of little concern, providing sets of criteria are carefully formulated and provision made in the research to evaluate them.

If practical problems, constructed according to identifiable guidelines or criteria, are to form the basis for empirical research, there are a number of further conditions which have to be met. In the first place, the problem situation and associated physical apparatus would have to be carefully constructed and their specifications clearly described. The examples in a previous section indicate how two practical situations, both seemingly involving identical aspects of measurement division and which could meet similar construction criteria, might stimulate two quite different sets of responses on the part of the children.

As long as it is clear under which set of conditions empirical data are to be collected, no great difficulty in their interpretation is likely to occur. Indeed, subtle changes in conditions and their effects on children's responses are among the most important issues this type of research is best designed to clarify. Distortions in interpretation would be almost certain to occur in the absence of precise information on the nature of these two situations.

Related to this requirement and of equal importance is the necessity of working out very precise protocols for presenting the problems to children. The context in which a problem is presented may have a profound influence on the way children respond to it. Whether the problem is presented to a group of children or to an individual child has to be clearly stated. Differences in language used to describe the problem, and suggestions or instructions about the form responses will take, are likely to be critical. Other considerations are whether one problem precedes another or whether specific instruction was given sometime prior to the child's response. Whatever verbalizations or directions are given to the child should be accurately reported along with an accurate report of the child's response. Seemingly slight (to an adult) changes in wording can significantly influence children's performance on tasks.

Whenever interviews are part of the procedure, it is clear that each would have to be tailored to the particular child and the way each responds in the situation. If the data consists mainly of children's spontaneous responses without intervention, it is important that protocols be established and that they be followed without substantial deviation from child to child. In either case, the necessity of carefully reporting precisely what was done and said cannot be over-emphasized.

It is important to keep in mind that the ultimate purpose of problem-solving research at this level is to obtain information which can be used to improve instruction in mathematics. The first step, however, is far removed from this ultimate goal. A great deal of information about how children respond to specific problems selected on the basis of specific criteria must be amassed before such problems have application in day-to-day instruction. The state of the art at this time would suggest that a clinical situation in which children respond individually should dominate the methodology. If time has to be spent developing this methodology, it should be considered part of the task we are facing. It has to be admitted that clinical research has not been developed to a great extent in North American mathematics education, and it is important that part of reports of clinical studies should be devoted to a discussion of the specific methodology used.

In the absence of detailed information about how children respond to practical, concrete problems in specific situations, it is difficult and probably undesirable to set up in advance a response framework or schedule. To overcome the difficulty a number of investigators have used audio or video tapes which provide reliable and faithful means of collecting data. No information need be lost and analysis can proceed as time permits using any number of schemes which show some promise of providing insight into the meaning of childish responses to problems. Each scheme can be applied simply by running the tape through again. The whole procedure provides rich data which requires the most competent, insightful analysis and interpretation.

Introducing (into the clinical situation) the technological devices necessary for such recording, however, produces its own peculiar problems. The situation which contains two or three video cameras with recorders along with the technical personnel required to operate them can be disturbing to children and may substantially alter their responses from those which might be obtained in a less busy atmosphere. Children will, for example, be attracted by monitoring devices and sometimes find it difficult to proceed with the problem, if they see their actions are being filmed. Measures taken to simplify the technical set-up and to ease the situation are as likely as not to result in an inferior record or incomplete data. In spite of these disadvantages some form of recording all of the responses of children seems to be mandatory at least in the early stages of such exploratory research.

One difficulty that cannot be over-emphasized is the expense connected with collecting and analysing audio and video records of children's responses. Even the most well-endowed investigator will become discouraged when analysing taped data. The reason for taping in the first place is that no adequate encoding scheme now exists which will select out all the important responses as they occur. But the flexibility and richness of taped data is at once a serious source of concern. A scheme or set of schemes, as often as not, have to emerge from the data themselves--which involves viewing taped segments over and over again. Otherwise, unexpected elements that influence children's thinking about a particular problem may be overlooked. For example, in a recent pilot study cages were provided and children asked to divide a number of animals equally among the cages. Their responses were video taped. The animals were mostly zoo animals among which was a single lion. It became apparent after several sessions that the lion was almost always left until last by younger children. Subsequent questioning revealed that the lion was too "dangerous" to put with other more benign types.

Not only is it difficult to devise and select an encoding scheme which will permit convenient analysis, but the sheer logistics of finding wanted tape segments in reel after reel of similar segments can be overwhelming. There are few investigators who have the tolerance required to encode taped material or to devise encoding schemes for more than two or three hours at a time with the level of alertness required by the task. Where research has to be conducted in the face of budget limitations it is important that the investigator make an accurate assessment of the time that analysis will take, and to adjust the amount of data collected accordingly. It is better to collect only those data which can be analysed with available resources than to collect large masses of data and hope that funds can be found eventually to analyse them. It is my guess that there are many hours of carefully collected taped data lying around right now waiting for analysis which will never be done because funds will never be available for the analysis. It is also my gues that funding agencies are turning down potentially good research involving taped data because their advisors or referees are ignorant of the power (and expense) of using the medium.

What has been said here about methodology in problem-solving research may suggest a rather narrow view of the scope of such research. It is admittedly a narrow view, but one which is taken to emphasize the need to make accurate observations of chilren's responses and behaviors when confronted with real, concrete, significant problems. We need to get a clearer picture of how children construct their own reality, what problem-solving abilities they possess at various levels and how these abilities develop with age and experience, what part spontaneous language plays in their constructions, how they interact with various visual and verbal stimuli in solving problems, and how their real experiences are used to build the mental structures we call mathematics. These and related questions have to be answered before we can confidently address the intricate instructional and curricular questions which is our ultimate task.

Until recently, research in all aspects of mathematics education has emphasized experimentation and the need to find a theory to account for learning phenomena. It has essentially skipped the phase which is purely descriptive and which depends on the ability of the investigator to make careful observations of children in learning situations. In their attempts to apply the methods of other sciences, researchers forget that development in disciplines such as medicine, agriculture and biology were preceded by years of simple observation and description. If we are going to make significant progress in research involving practical problems it is essential that the phase which is characterized by observation and description precede serious attempts to experiment or to develop a theory.

## Some Outcomes of Research Involving Real or Practical Problems

This section will be devoted to describing certain aspects of two lines of research involving practical problems currently being conducted at the University of Alberta. The main procedures and methodology of both emphasize observation and description. While they do not define the scope of such research, they do provide samples of a kind of methodology that shows promise.

The first project was designed by Nelson and Sawada, and is concerned with responses of children in the age range three to nine years as they were presented with a selection of six practical problems paired with six others. The pairing of problems was arranged so that the physical situations in which each pair occur were dissimilar while the mathematical structure on which each pair was based was the same. Problems involved the following mathematical processes or notions: a) division (measurement and partitive); b) locating positions in two and three dimensional space; c) sequences; d) geometric constructions; e) predicting movement in a plane; and f) factoring.

To give an idea about the nature of the problems, those involving location in two and three dimensional space will be described briefly. The problem in two-dimensional space was a simulated parking lot drawn on a large board painted black. Parking stalls were drawn in with white lines. Small plastic cars about 3 cm long were provided for the child along with a card on which two numerals were written. The first numeral indicated the number of spaces to go in from the entrance and the second the number of spaces into the lot to park the cars. Coordination of these numbers would permit the child to park correctly. Those who parked at least one car correctly were then asked to pick out the car from among three which indicated where one of the correctly parked cars could be located.

A parallel or equivalent problem was a simulated theatre with three decks or balconies. Each deck was painted a different color. Rows of seats were designated by letters and seats in a row by numerals. The child was given a small child-like figure and a card and asked to find the seat for the figure. Cards were made to match the color of the decks and on these cards was given a letter and a numeral. Again, children who seated at least one figure correctly were asked to select the card from among those which showed where the figure was located. Criteria for the construction of these problems have already been listed.

Sampling of responses was arranged to account for development of responses across the age range with longitudinal verification after one year. Sampling procedures also took into account the effect of order of presentation of a problem and its equivalent.

> The cross sectional data were made up of responses of each child to six different problems. These responses were all recorded on video tape in one setting. Sampling of the problems was so arranged that ten different children at each age level did each basic problem while five of these also did its equivalent. The same pattern was to have applied in collecting the longitudinal data a year later except that now ten children at each age level were to do the equivalent problems while five of these were to do the basic one. Normal attrition reduced these numbers slightly but not enough to do serious harm. (Nelson & Sawada, 1975)

Data were subsequently analysed by viewing the taped segments of each child who responded to a particular problem or its equivalent. No prearranged coding scheme was designed in advance. Schemes were allowed to grow out of the observations. Descriptive accounts of the results have appeared in several publications (Bourgeois & Nelson, 1977; Nelson, 1976; Nelson & Kieren, 1977; Nelson & Sawada, 1975). Analyses for some of the problems have not yet been completed.

The other project to be considered was designed by Kieren and was preceded by a careful analysis of possible interpretations of rational number construct. Kieren identified seven interpretations for fractional and rational numbers:

fractions
decimals
ordered pairs (equivalent classes)
measures
quotients
operators
ratios

The cognitive and instructional structures required for building a rational number construct as suggested by Kieren are: part-whole relationships, ratios, quotients, measures, and operators. For each of these, a set of tasks or problems can be devised appropriate for children learning the construct.

The same criteria for constructing problems used by Nelson and Sawada were used by Kieren to gather information on the child's notion of rational numbers as operators. The operator notion is based on mechanisms which map a set (or region) multiplicatively onto another set. (A "3 for 4" operator would map a domain element 16 onto a range element 12 while a 3/4 operator maps a region onto a similar region reduced in size.)

The practical problem consisted of a card-stacking machine whose input could be compared with an output to define the nature of the operator. Observations of forty-five children in the age range 8 years 11 months to 14 years 7 months and descriptions of their responses to these kinds of situations have been summarized by Kieren and Nelson (1978).

Here again no coding scheme was developed in advance but grew out of the observations of children as they solved problems in the situations as outlined. However, it should be pointed out that the conditions and experiences were carefully designed so that responses to them could be readily observed.

With no statistical hypotheses to reject, what is the form that reports of such observations take? Can the results be used as a basis for more formal experimental research? Are new insights into childish behavior possible with these methods? To help answer these and related questions, some results which have so far been obtained will be reported here.

In both the studies outlined, problems were embedded in physical layouts to which children could respond in a physical way. The Kieren study involved interviews requiring verbal and symbolic responses. The protocols developed for the Nelson and Sawada study provided support in case children did not respond, but the precautions were found to be unnecessary and were not included in the longitudinal sampling. In most cases children were not only prepared but were eager to respond in a physical way. This phenomenon was no more apparent in younger than in older children even though the older ones may have been able to respond symbolically. Anyone doing research involving real, practical problems need have no fear of any reluctance on the part of children--even the very young--to respond eagerly and, for the most part, in a readily interpretable manner.

Spontaneous language used by the children was of considerable interest. There was, for example, in the Nelson and Sawada study a noticeable change in language function across the age range. Whether language was being used to help solve the problem or whether the problem proved to be a useful source of language generation by the child could not be determined, but younger children used language extensively to monitor their actions. In fact, with three- and four-year-olds particularly, the language often defined some problem other than the intended one. Older children, on the other hand, used language to pose questions in order to clarify more precisely what problem they were expected to solve. Few children older than five altered the problem to suit themselves.

Provoked language, as in the exploratory study reported by Kieren and Nelson, can provide rather clear insights into children's modes of thought in dealing with problems. When asked to describe how they thought the fraction machine functioned, for example, it was clear that many children thought subtractively and not multiplicatively. For example, in looking at the 2/3 operator such children would say it's subtracting 4(12→8), it's subtracting 10 (30→20), and thus never focused on the constant multiplier involved.

Interpretation of language function may have been improved in some instances if an expert in the language development of children had been part of the investigating teams. Those proposing to do research in the problem-solving area would do well to recruit such a person in the early planning stages.

Kieren is in the process of exploring in greater depth the role of the operator in the development of the rational number construct in children. The main thrust of this research will be to investigate more thoroughly the tendency of children to think subtractively rather than multiplicatively when working with operators, and to look at the partitioning act as a vehicle in partitive performance. It should be noted that, in the exploratory study, nearly all children mastered the 1/2 task; but when faced with the 3/4 task on the machine, the vast majority of children under 12 would give 12 as an output for an input of 24. Questioning revealed that they knew it was not a 1/2 machine but when confused would respond as if it were. The global role of 1/2 in early thinking obviously needs careful investigation.

A third function of Kieren's exploration will be to trace the developing ability of children to move from functioning with unit operators to functions with all forms of operators. The method will be clinical and will emphasize careful observations and descriptions of how children respond to protocol problems involving operators.

In the Nelson and Sawada study there were twelve problem situations (or more precisely, six pairs of problem situations). In general, these took the form of layouts or materials which children could manipulate in order to solve associated problems as in the example given. As pointed out before, there was no reluctance on the part of children to respond--but there were a number of other observations that applied to more than one problem situation. For example, children were often distracted from making appropriate responses to problems because of various spatial and physical characteristics of the situations. For example, the division problem required that children load colored plastic cars on a simulated ferry boat three at a time and tell how many trips it would take to get a group of cars (15) across the simulated river. Some children tended to focus on the color of the cars and wanted to load all cars of a particular color at once. Since the grouping based on color was different from the required grouping they failed to get the required result (Bourgeois & Nelson, 1977).

In a partitive division problem it was necessary for children to park cars in front of simulated houses so that there would be the same number of cars in front of each house. Some children would not park all the cars because that would be "too many" cars for each house. Or they refused to park cars on the "grass" near the house. A three-year-old was so interested in the make and model of plastic cars used in the parking lot problem (locating positions in two dimensions) that he forgot the rules given for parking.

The vulnerability of children to such distractions is not new. It is evident in the non-conservation behavior of children. Children who cannot conserve (whether it be number, length, area, volume or whatever else) cannot do so because of some irrelevant or distracting element in the situation to which these children respond. The point is that if we are going to provide practical problems for children, we have to be able to predict with some confidence (as in the conservation phenomena) what may be distracting in the problem situation and thus interfere with the child's ability to cope with the problem.

The information from the Nelson and Sawada study indicating the tendency of children to be distracted--or, more precisely, to respond to distracting elements of the situation--has led to more searching clinical study with this phenomenon (Bana & Nelson, 1977; Bana & Nelson, 1978). Although the work is far from complete, these studies have revealed some interesting results. Kieren is finding, for example, in the machine problem that children preserved their own answer by using completely inconsistent explanations. The necessity of justifying their answers appears to be so distracting that logic and consistency is overpowered. Bana and Nelson have found that children seem to have a greater tendency to be distracted if, when the distracting element is brought into play, it provides a plausible alternative problem for the child to solve. There is also some evidence to support the contention that the way a problem is posed can determine whether a child will be distracted or not. Whether these two observations can be verified, and if so whether they are in fact part of the same difficulty, depends on further carefully designed clinical research with appropriate problem settings.

Whenever children in the Nelson and Sawada study were required to predict an outcome, there was a distinct reluctance on the part of many to attempt to do so. In fact, nearly half of the children across the age range refused to predict without considerable urging. The proportion of those who were reluctant to predict showed little change from three to eight years. The same phenomena shows up in the Kieren study as these older children also appeared more happy to say nothing than to be wrong. It is not clear at this point what the true dimensions of this pheonomenon are. If it were school-induced it is not likely it would manifest itself so strongly in pre-school children.

There are some specific outcomes which warrant mention here as examples of the kind of information research involving real problems is likely to reveal.

It is generally conceded that partitive division is a more difficult process for young children than measurement division. In any case, making groups of a specific size can be more easily systematized than partitioning a given number of objects into smaller groups. Children in the Nelson and Sawada study had no completely systematic way of partitioning and generally found these partitioning problems more difficult except when in measurement division no provision was made in the problem to preserve the integrity of the equal groups. Thus, when animals were placed in cages, children had no difficulty saying how many cages were needed. But, when the ferry boat had finished hauling cars across three at a time, children had trouble remembering how many trips the ferry took (Bourgois & Nelson, 1977). This is the same difficulty with measurement division that was discussed earlier.

Even very young children were successful in constructing complicated three-dimensional figures when provided with a number of two-dimensional elements (Nelson & Kieren, 1977). Although they appeared in many cases to be solutions strictly on the physical level, providing little or no mathematical-logical experience, such problems seemed to be appropriate for the whole age range three to nine. What effect such early experience has on the subsequent development of spatial abilities in children is yet to be determined. Their skill in making structures and their eagerness to do so suggest that the effect on these abilities may be considerable. The relative success of very young children in some of these tasks was no doubt, the result of more or less favorable modes of presenting the problems.

A pair of problems was designed to determine if any children in the age range three to nine relate numbers and their factors. One problem was called the factor platform. This was an upright structure slightly sloping backward with thirteen slots and blocks which could be piled in the slots. Children were presented first with twelve blocks in four of the slots arranged so there was not the same number of blocks in any two slots. They were asked if the blocks could be rearranged in the four slots so there were the same number in each slot. This proved to be easy to verify for almost all the children (some three-year-olds piled all the blocks in one slot) but few if any thought of twelve blocks being arranged in four groups of three.

When one block was removed so that there were now a total of eleven and they were distributed in four slots, again so no two slots contained the same number of blocks, most of the younger children persisted in trying to arrange them in equal piles. Their failure to do so did not, in any case, suggest to them a difference in factorability of eleven and twelve. This was expected to provide only physical experience for the three-, four-, five-, and six-year-olds, but it was expected that at least some of the older children would suspect what was going on. Experience with factor boards which had spaces for blocks to fit in twos, threes, and fours did not make it any easier for children to see in advance that eleven blocks could not be made to fit exactly in any of them. The inappropriateness of this set of problems to reveal anything of importance is in sharp contrast to the other problems included in the study. Nine-year-olds in the longitudinal sample who had been in school as much as four full years could have been expected to respond more appropriately to these situations if any instruction at all had been provided in school to partition the set of counting numbers. Either that, or the notion of partitioning according to factorability of numbers is too complicated for eight- and nine-year-olds.

The examples given above serve to illustrate the kind of outcomes that can be expected in clinical methods involving real problems. While most of the observations need further clarification and more rigorous verification, they do form the basis of a methodology which promises profound insights into the way children go about solving problems.

Despite the crudities in methodology, the studies cited in previous sections lend support to the following general conclusions:

1. Distraction appears to be a key element for children dealing with practicial problems. It is manifested in the form of responses young children make to various irrelevant physical, spatial, and numerical aspects of the problem situation. It also occurs in a somewhat altered form in older children who are so attracted to justifying their own answers that they cannot give logical explanations for the mathematical procedures involved.

2. Children can get involved in more elaborate mathematical processes when they are embedded in relevant, practical problems than when the same processes are presented in their more formal, abstract or symbolic forms. Thus three- and four-year-olds, though not necessarily in a perfectly systematic way, find solutions to partitive division problems with real objects while ten-year-olds perform the complicated partitioning required in handling compound fractional operators (multiplication of fractions) provided the process is embedded in the card-stacking machine. What this means in terms of instruction is not altogether clear, but children seem to be able to "act out" mathematical processes in real problems long before the same processes make any sense at all in the symbolic forms.

3. Children involved in solving real problems are more apt to engage in a genuine search for solutions. This stands in sharp contrast to their responses in solving verbal problems where there is a search of sorts but that search is for a formula or a procedure which can be applied to produce desired answers.

4. Careful observations of children solving real problems provides a brighter picture of the interface between their development and their experience. Distraction, for example, seems to occur more as a function of being able to formulate a plausible alternative problem to the one intended than of how complex the problem is. In fact, complexity does not appear to be an important factor in whether or not a child will be distracted.

Other general conclusions could be drawn, but these four are illustrative of how rich the field is or can become.

Nelson

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# Applied Problem Solving: Models

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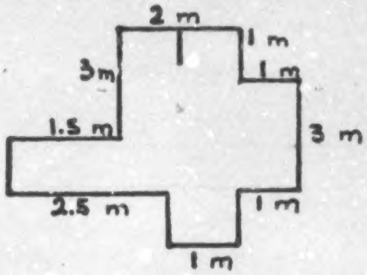
## A Setting

The Nelsons wish to carpet a small room of an irregular shape. In doing this they want to estimate the amount of carpet they will need to purchase.

This is an ordinary problem in applied mathematics. The question is, what will the Nelsons, what might anybody, what could anybody do in the face of this problem? Of course, if the room were a rectangle 3 metres by 4 metres, the problem would be limited. Most persons, almost unconsciously, would apply the model "area is 3 x 4 square metres." Or they might say "since a roll of carpet is 4 metres wide, we'll need 3 metres." The former model simply applies a well-known generalization in the sense of simply using the particular numbers involved. The latter model is a simple "count-match" model.

This "ordinary" problem is seldom so ordinary that the modelling behaviours above would make a perfect fit. So the question remains what do persons do in the face of such a problem? Probably, the Nelsons would draw a picture or a scale drawing of the floor they wish to cover.



Moving from a rough sketch to a scale drawing involves the conscious use of proportional thinking. At this point two other modelling strategies might come into play; rectangular decomposition or the use of a grid. The use of either of these would again be based on the availability of mathematical ideas (definition of area) and perceptual and constructive mechanisms.

The application of the above models still may not solve the problem at hand. There is the problem of relating the floor area (a number) to the number of square metres of carpet to be purchased off a 4-metre-wide roll (a number which will hopefully be close to the floor area). This latter problem again requires modelling behaviour which includes minimizing waste. This problem could be confounded if one were buying striped carpet, (Which direction of laying it will yield minimum waste? Give the best appearance?) The second parenthetical question suggests that modelling in applied settings often must take into account a variety of criteria--aesthetics, taste, quality, colour, personal relationships.

To conclude this illustration the carpet salesperson calculates the amount of carpet needed and gives a price at $7.50/metre. The Nelson's then must compare their estimate with that of the salesperson and evaluate both their procedures and the outcome.

It was the purpose of the above story to illustrate some of the complexities involved in discussing modelling behaviours used in applied problem solving. It is the purpose of this essay to describe several models of problem solving and to draw from them ideas about modelling behaviour. In particular, the paper will then try to look at the personal mechanisms and knowledge necessary for applied problem solving. An attempt will be made to discuss a general cycle of growth of such knowledge and mechanisms. All of the foregoing purposes find ultimate meaning in the question of developing curriculum and instructional environmental for the development of applied problem solving.

## Models of Problem Solving

### 2.1 Sources of Models

Griffiths and Howson (1974) suggest that a curriculum developer must consider the interests of four domains in building an effective set of mathematical learning experiences. These are elaborated in Figure 1 below.

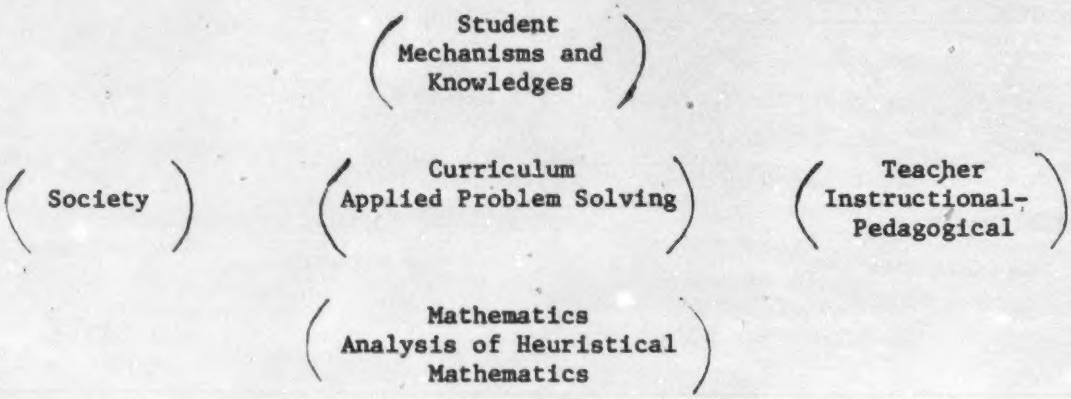


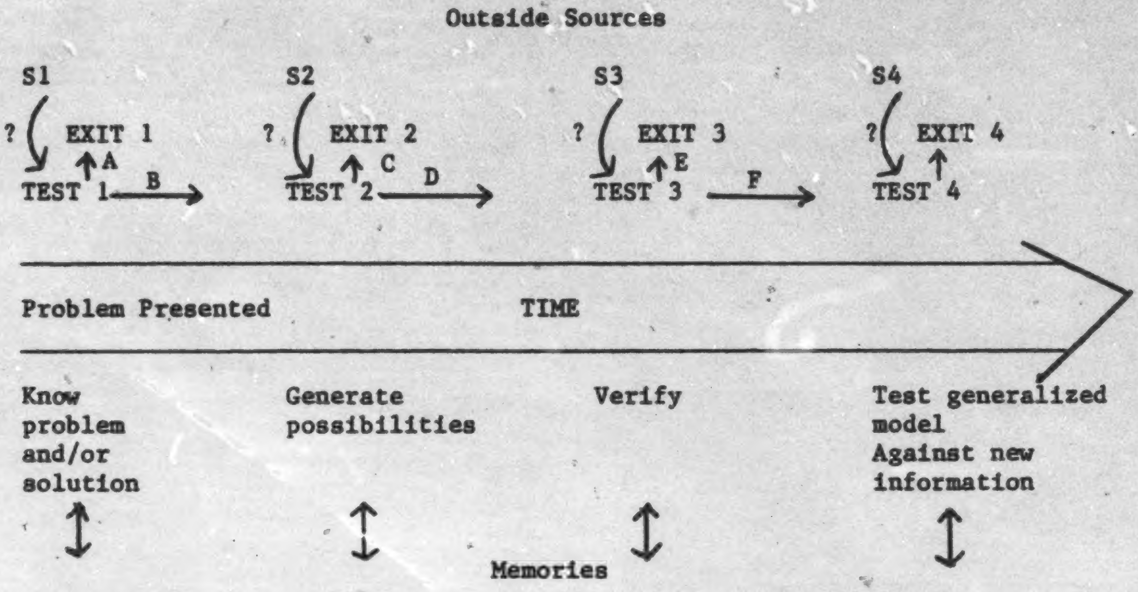
Figure 1

Each of the four influential components, students, teacher, mathematics, and society, can also be interpreted as influential in the development of experiences in applied problem solving as descriptive sources of useful modelling behaviours.

### 2.2 Personal Sources

Over the past century there have been numerous models of the problem-solving process as seen in individuals. Classically, these models had phases such as "incubation" and "illumination." These structures as posed by

Dewey (1910) and Wallas (1945) stimulated the imagination and thinking of theorists, curriculum developers, and teachers alike. Guilford (1967) gave a scheme which looked at the problem solving process as a system and incorporated many of the earlier theorized processes.



There are several virtues in this model as one considers applied problem solving. First, it clearly envisions roles for both internal and external information sources and their coordination. Second, it demonstrates that a person may choose to stop the process at many points either giving up, being distracted by a side issue or possibility, or through satisfaction with an achieved status. Yet the model is not closed, indicating that problems, particularly those of an applied nature, have the potential for being enlarged, changed, or elaborated.

As yeasty as the above model is, it presents a person developing experiences for students with some problems. First, how does one implement progress through the model? Second, what kinds of (modelling) behaviour does a person use in applied problem solving in mathematics? Boychuk (1975), pursuing earlier work by Evans (1965) and Taylor-Pearce (1970), interviewed grade 9 students as they worked on problems designed to allow and indeed elicit certain problem solving mechanisms. These general mechanisms were seen as sensitivity to the problem, redefinition, conjecturing, and verifying. Although these four behavioural categories can be further detailed, they still give us only limited insight into the modelling behaviour of the problem solver. However, one important result is seen. Performance on these specific mathematical tasks is seen by both Boychuk and Taylor-Pearce to have a very low correlation with more general "creative behaviour" performance. Thus, one might conjecture, that, in looking for knowledge about modelling, one is well-advised to be concerned about mathematical settings and techniques.

## 2.3 Mathematical - Instructional Sources

Of course, as one thinks of sources of problem solving behaviour, one is drawn to the classic work of Polya (1957). Many of his heuristics have application in applied problem solving. Because applied problems are usually complicated, the solver frequently reduces them to a simpler (or similar) known problem and is frequently at least temporarily satisfied with the solution to this problem.

LeBlanc (1977) has elaborated an instructional model for problem solving. This model is rich in examples of modelling behaviours in problem solving. As a person constructs a (mathematical) model in solving an applied problem, the acts of diagramming, systematically listing information, and making tables of known values are of value. In fact, any one of them might serve in and of itself as a mathematical model for a problem. Not only are such devices models but they are also forms of representation. As such they might trigger in the mind of the user different mathematical thinking or models as well.

In the initial example in this paper, it was suggested that one modelling behaviour used was to try to find a "formula" which works, that is, which models the problem. Robinson (1977) has elaborated on the use of formulas as models. She focuses on how one constructs models through the transformation of known information or formulas, as well as finding information in the problem setting which will allow for the use of a given model. Thus if one needs the altitude of an object fo find its surface area and this is not attainable directly, can one substitute some other known value effectively? Can one deduce the needed information from the given information (e.g., use the Sine Law)? Can one alter one's formula so as to use only knowable values?

It is evident from the discussion above that such modelling involves knowledge of mathematical generalizations. How does one come to use such generalizations in modelling? Certainly one way is through the possession of genius and its application (Hadamard, 1945). While it is true that possession of unusual mathematical abilities is rare, it is also the assumption of the above discussion that most persons can be taught or at least given experiences with the heuristics suggested. Thus it is hoped that through the use of such heuristics most persons could model real problems at least on a limited scale.

## 2.4 Societal Sources

Since applied problem solving occurs in the "real world," it can be assumed that there would be numerous observable examples of modelling. One problem with such observation is that the observer must know a lot about the field being observed in order to abstract mathematical models of even mathematical modelling behaviour.

Certainly one area where mathematical modelling in applied problem solving is very visible to anyone is in the various uses of computing devices and information processors. In fact, such devices and the underlying model are ubiquitous in North American society.

There are four aspects of informatic thinking which are sources for modelling behaviours. The first and most important is algorithm design. Such design involves developing a definable procedure for accomplishing a given task. Such procedures may involve closed formulae or they may involve recursion where the result at one stage in the process is based repeatedly on the result at the next prior stage. Such algorithms can define the use and sequencing of information as well.

A second element of algorithmic thinking is algorithm application. In computing device terms, this means devising a sequence of steps which allows a machine to execute an algorithm. This sequence (program or key stroke sequence, for example) like the algorithm above represents a model of some process or problem to be solved. This kind of modelling behaviour (programming) has three important features. First, it is personal although following a specified syntax. Second, it can be communicated in a standard way to others. Third, it is directly testable (one executes the program).

Data organization is a third element of informatic thinking. Here the person anticipates the output from executing a program and builds into it a way in which this output should appear. Thus values for related variables might be positioned so relationships might be seen, or it might be simply structuring an output message to give information compactly ("The flights to Washington are at __, __, ___"). This organized data might contain a model for a problem and the act of organizing data is in a sense a modelling behaviour.

The fourth informatic thinking element is "data application." Here the person uses the output generated to solve some problem. This act of relating information back to the original problem, while the culmination of the problem solving activity, serves to test the modelling done in the previous three activities. If the prior models were effective, this last stage should be greatly enhanced.

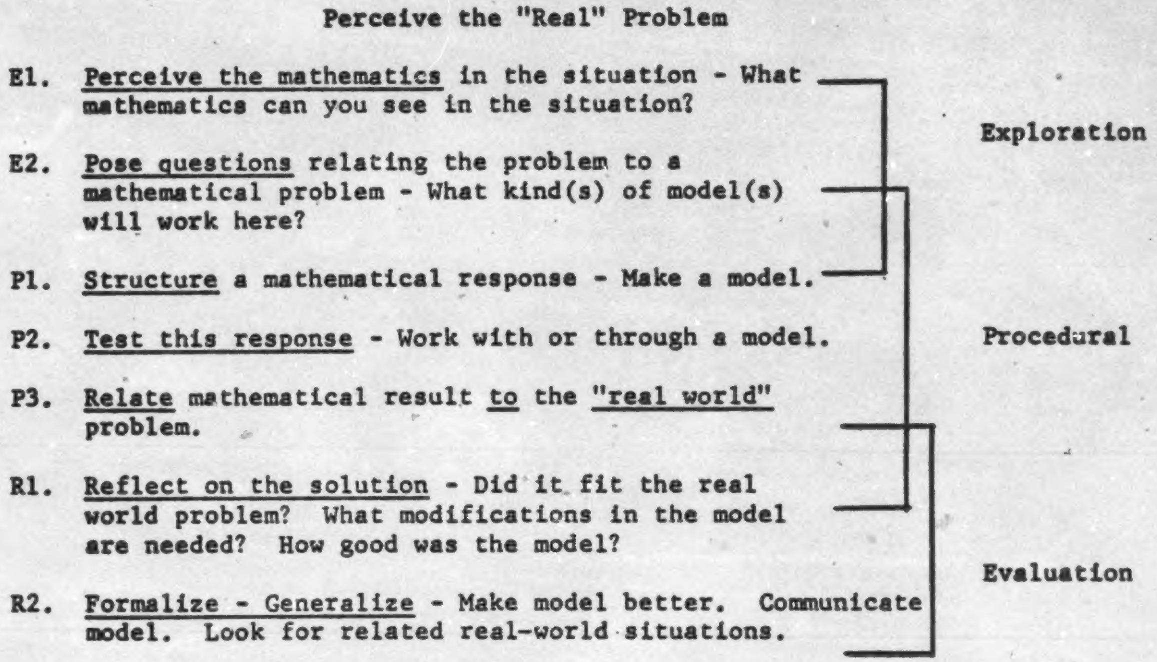
As described above, informatic experiences provide a rich source of modelling behaviour. To be sure, algorithmic thinking and its related behaviours do not apply to all problems. Yet with the advent of sophisticated micro-processors with various graphic capabilities, such thinking and its use are available to a large number of people.

## Curriculum and Research Hypotheses

### 3.1 A cyclic model

Section 2 above presented models for problem solving derived from psychological, mathematical, and societal sources. From these models certain inferences were made for modelling in applied problem solving. These models and thus the inferences had two things in common. The moves within a model were heuristic in nature and occurred in a sequence characterized by exploration of elements, followed by execution of a procedure, followed by evaluation or elaboration. Supporting this sequence of heuristical moves lay knowledge of useful mathematical generalizations. Thus it seems that experiences in applied problem solving and research about modelling behaviour should take these commonalities into account.

The following three-phase scheme for discussing applied problem solving is an adaptation of a model for mathematical knowing developed by Sigurdson (Note 1). The three phases of exploration, execution, and evaluation over-lap indicating interdependency as well as a certain amount of concurrency among them.



Perceive the "Real" Problem

E1. Perceive the mathematics in the situation - What mathematics can you see in the situation?

E2. Pose questions relating the problem to a mathematical problem - What kind(s) of model(s) will work here?

P1. Structure a mathematical response - Make a model.

P2. Test this response - Work with or through a model.

P3. Relate mathematical result to the "real world" problem.

R1. Reflect on the solution - Did it fit the real world problem? What modifications in the model are needed? How good was the model?

R2. Formalize - Generalize - Make model better. Communicate model. Look for related real-world situations.

Figure 2. A scheme for applied problem solving with reference to modelling

This scheme suggests many instructional hypotheses. In the exploration phase as well as P1 a person has to rely on certain mathematical background brought to the situation and particularly on abilities such as redefinition of situations and generation of possible alternatives. In addition, the skills or algorithm design and design of useful lists and tables of information would prove helpful.

The procedural phase makes three modelling demands on a person. First, a person may need to transform a known model to fit the situation. Second, this model must be made operational. One's plan must be made to generate possible solutions to the problem. Finally, one must be able to transform one's mathematical solution into a practical one. This latter, will probably demand knowledge about the area to which mathematics is to be applied. (e.g., What does it mean to get a negative profit? A negative maximum height for a rocket flight?)

The evaluation phase requires other modelling skills. One is the ability to define the key features of two models and test a relationship between them. (The mathematical model $A = bh$ may depend on the slant height of a building, when the information available is the altitude.) A second modelling ability, which may also be needed in P1 and P2, is the making of an effective symbolic model and deciding on some standard form for it.

Thus, applied problem solving requires one to have knowledge of "models" in the applied field, and the knowledge, cognitive abilities, and heuristics to see the mathematics in the model and generate a mathematical problem. Procedurally one must, through the use of algorithmic thinking or heuristics discussed earlier, design a model and put it into functional form. Mathematical solutions must be transformed into problem solutions. In the evaluation phase one must be able to perceive isomorphism between mathematical and applied settings as well as between the form of the particular procedure and a possible symbolic standard.

The scheme represents a system in a number of ways. Each of the phases provides either input or feedback to other phases. The model and its execution in the P phase provides feedback about one's mathematical perception in phase E. Finally, the scheme reveals a cyclic nature because of the exploratory aspect of generalizing the use of the model to other situations.

### 3.2 Research Questions

The scheme for applied problem solving described above has numerous implications for research on modelling. One set of questions is developmental in nature: How do various modelling behaviours manifest themselves at various ages? That such questions are even reasonable can perhaps be seen in the responses of two children ages 5 and 8 to the following "applied" problem:

"Here are some boxes (15 in 4 colors). Load them on the trucks (3 which each had spaces for 6 boxes) so that each truck has the same number on it."

The five-year-old perceived "same" and recalled a "dealing out", or partitioning, model. He successfully dealt one box to each truck until he exhausted the set.

The boxes were dumped and the child was asked if he could solve the problem another way (another model). "Well," he said, "you could deal them out in the other direction (left to right) but that's just the same."

It is interesting to note that this child did not check his result for the number on each truck, nor did he worry about number in the original set. Other young children faced with this problem perceive other mathematical elements (e.g., match boxes to places on the truck or simply place one on each truck and quit) or react to other elements (e.g., color) in the problem.

The eight-year-old also dealt the boxes out one by one. She announced that each truck had five (as if to check her answer). In a second trial she modified her "model" dealing out three at a time in an attempt to do two things--attain an equal number on each truck out to generate a different numerical result (six per truck). After much effort and some mental calculations she finally convinced herself that she would always get five if the numbers were to be the same.

What is germane from the above example is that the modelling behaviours of young children (at least) are observable. Further, the partitioning model of the eight-year-old was different from that of the five-year-old, the former being more general (by ones, threes, etc.) and also being seen as related to number. Also, these models were only two among many used by children in similar situations.

A second set of research questions deals with the relationships between various student characteristics and modelling skills in the various phases. For example, how do measurable abilities such as mathematical creativity, skill at redefinition, or field dependence relate to applied problem solving and particularly to mathematical modelling behaviours exhibited? How does achievement of particular mathematical generalizations, particular heuristic skills, or certain constructive mechanism capabilities relate to various models selected and used by persons in various situations and at various ages? For example, how do measurable behaviours in partitioning, pattern finding, or formula transformation relate to the ability to structure, execute, or symbolize a model in an applied problem solving situation? A related question set would try to ascertain the relative effects of mathematical knowledge, heuristic skill levels and applied subject field knowledge on applied problem solving.

A final set of questions deals with instruction in applied problem solving. One kind of research would try to ascertain which abilities, heuristic skills, and constructive mechanisms were amenable to instruction. Particular instructional programs could be devised and their outcomes tested using applied problem solving abilities and modelling in particular as criterion measures. A second kind of instructional research would look at the effects of various curricula in applied problem solving on the various modelling behaviours noted above. One example of such research could test the effect of the representation of the problem (in how real or concrete a form) on the kind of models used by persons solving the problems posed. Kieren and Southwell (1979) found that subjects faced with a concrete representation of a problem requiring fractional operator thinking used analytic models, whereas a significant proportion of subjects faced with the same "mathematical" problem in a symbolic form used algebraic models in solving the problem. Because such instruction might be highly dependent on the teacher and on actual student activities, observational research relating planned instruction with teachers' reflections on various acts with related student modelling activities would also be fruitful.

Mathematical models are considered important tools and objects of study in mathematics itself and of course in many applied fields. However, little is known about mathematical modelling as a human activity. It is hoped that research pursued along the above lines will allow for better knowledge of this activity, how it develops and how it can be fostered. Such knowledge would be important in relating present knowledge on mathematics learning and abilities to the important area of applied problem solving.

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